CHAPTER 6

Risk Aversion and Capital Allocation to Risky Assets

McGraw-Hill/Irwin

Allocation to Risky Assets

- Investors will avoid risk unless there is a reward.
- The utility model allows optimal allocation between a risky portfolio and a risk-free asset.

Risk and Risk Aversion

- Speculation
  - Taking *considerable* risk for a *commensurate* gain (a positive risk premium)
  - Parties have heterogeneous expectations and assign different probabilities

Risk and Risk Aversion

- Gamble
  - Bet or wager on an uncertain outcome for enjoyment
  - Parties assign the same probabilities to the possible outcomes
  - A *fair game* (zero risk premium) is similar to gambling. A risk averse investor will reject it.

Risk Aversion and Utility Values

- Investors are willing to consider:
  - risk-free assets
  - speculative positions with positive risk premia
- Investors will reject fair games or worse
- Portfolio attractiveness increases with expected return and decreases with risk.
- What happens when return increases with risk?

Table 6.1 Available Risky Portfolios (Risk-free Rate = 5%)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

How to compare?
Each portfolio receives a utility score to assess the investor's risk/return trade off

A Utility Function

$$U = E[r] - \frac{1}{2} A\sigma^2$$

U = utility or welfare (measure of happiness)

E[r] = expected return on the asset or portfolio

A = coefficient of risk aversion

σ^2 = variance of returns

½ = a scaling factor

There are other utility functions out there: must increase with *E[r]* and decrease with σ^2

A Utility Function – Meaning of A

$$U = E[r] - \frac{1}{2} A\sigma^2$$

A = coefficient of risk aversion. Interpretation:

- **A>0: Risk Averse**. Penalizes risk. Will want a larger risk premium for riskier investments.
- **A=0: Risk Neutral**. A pure trader, only concerned about expectation. Will accept a fair game.
- **A<0: Risk Lover**. Adjusts utility up for risk because enjoys the risk. A gambler, bored with risk-free, will prefer for riskier investments.

Table 6.2 Utility Scores of Alternative Portfolios for Investors with Varying Degree of Risk Aversion

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

$$U = E[r] - \frac{1}{2} A\sigma^2$$

Three investors with A= 2.0, 3.5 and 5.0
Q. What portfolio will each choose?

Investor Risk Aversion (*A*)	Utility Score of Portfolio *L* [$E(r) = .07$; $\sigma = .05$]	Utility Score of Portfolio *M* [$E(r) = .09$; $\sigma = .10$]	Utility Score of Portfolio *H* [$E(r) = .13$; $\sigma = .20$]
2.0	$.07 - ½ \times 2 \times .05^2 = .0675$	$.09 - ½ \times 2 \times .1^2 = .0800$	$\mathbf{.13 - ½ \times 2 \times .2^2 = .09}$
3.5	$.07 - ½ \times 3.5 \times .05^2 = .0656$	$\mathbf{.09 - ½ \times 3.5 \times .1^2 = .0725}$	$.13 - ½ \times 3.5 \times .2^2 = .06$
5.0	$.07 - ½ \times 5 \times .05^2 = .0638$	$\mathbf{.09 - ½ \times 5 \times .1^2 = .0650}$	$.13 - ½ \times 5 \times .2^2 = .03$

Table 6.2

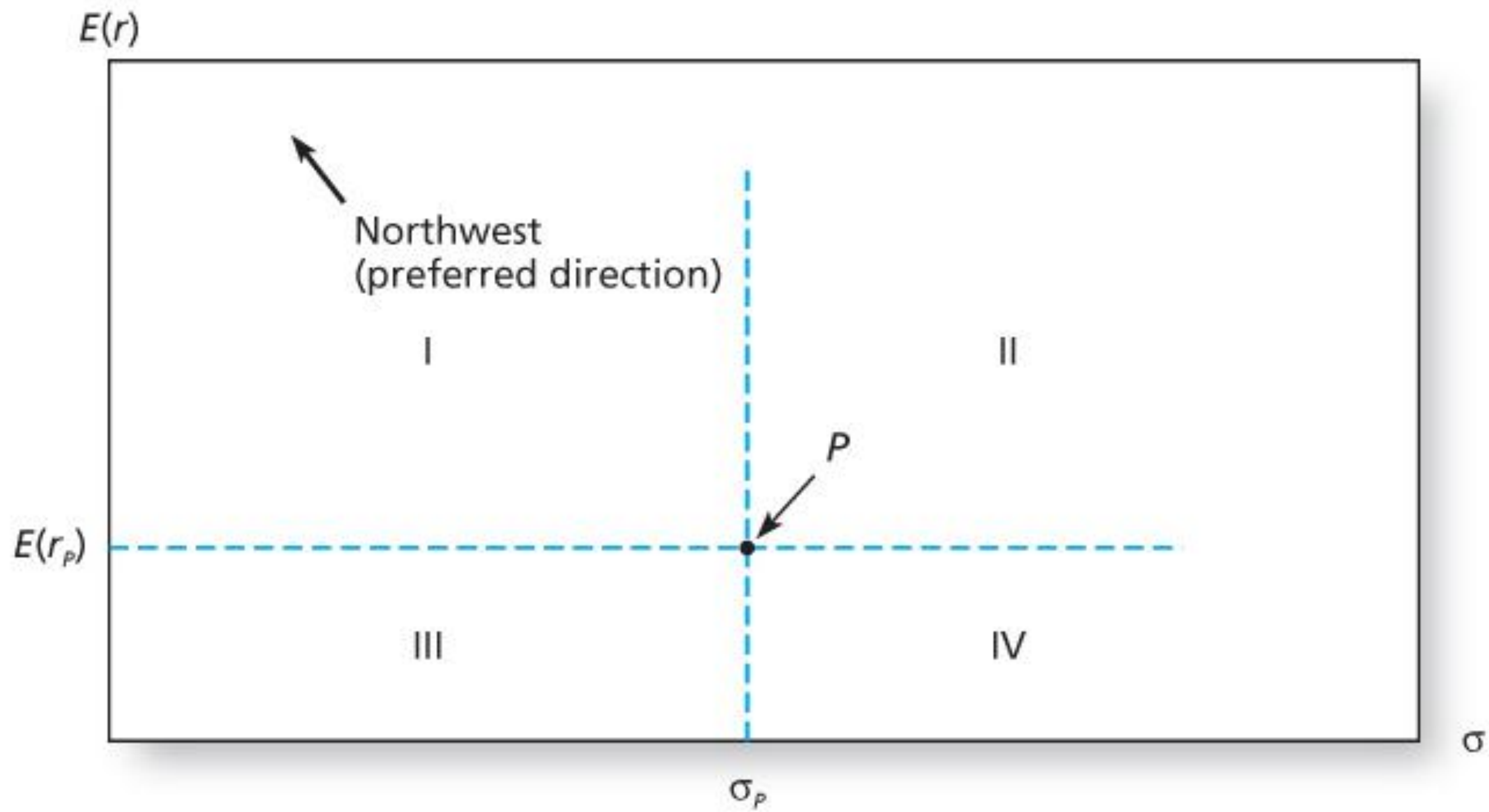
Let's play a game

- I will toss a coin and pay you some money X if heads and nothing if tails
- How much are you willing to pay to play this game?
  - For X=$0
  - For X=$1
  - For X=$10
  - For larger X?

Let's flip the game

- I will toss a coin and *you pay me* some money X if heads and nothing if tails
- How much are you asking *me* to play this game?
  - For X=$0
  - For X=$1
  - For X=$10
  - For larger X?

Risk-Return Trade-off



Mean-Variance (M-V) Criterion

- Portfolio A *dominates* portfolio B if:

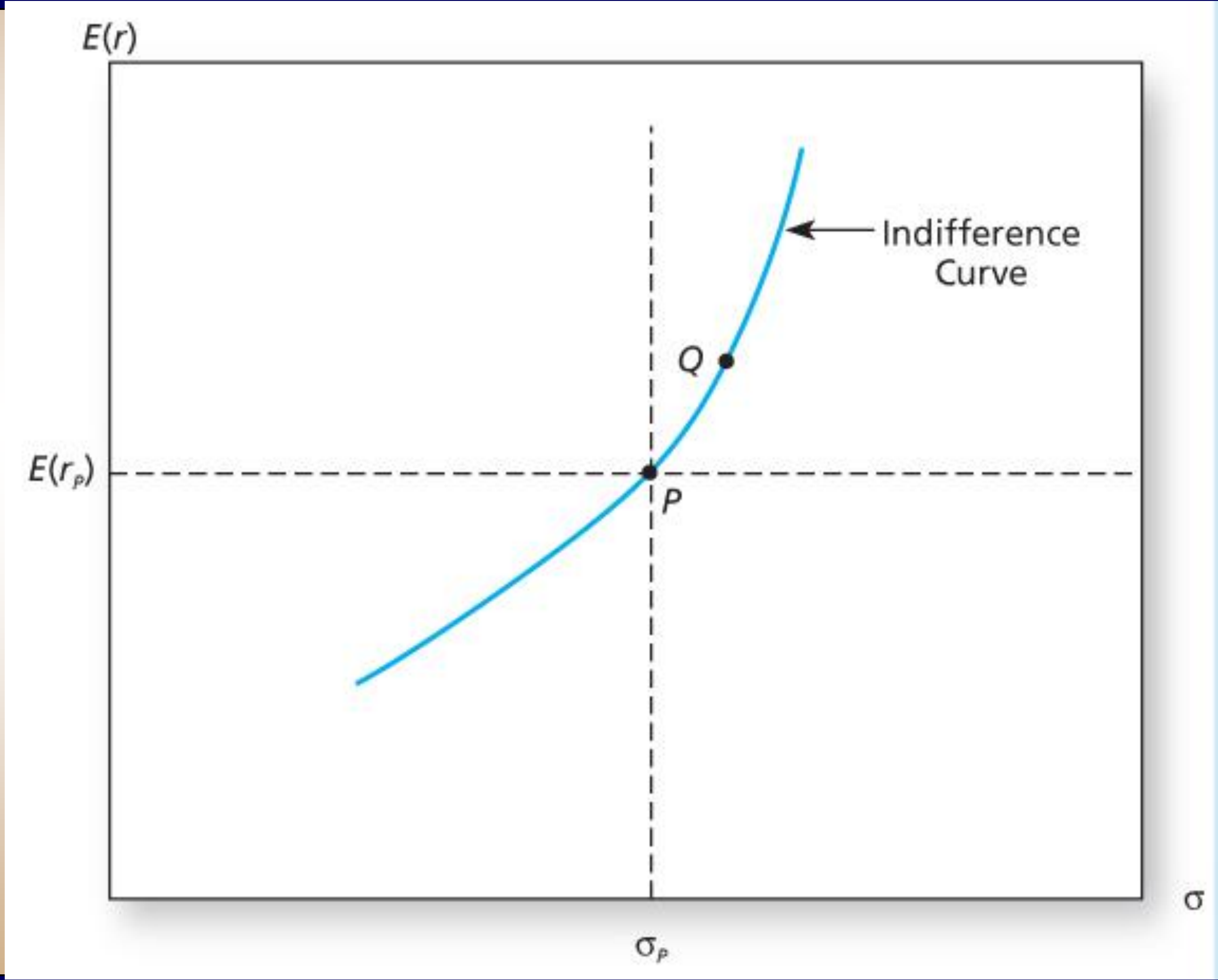
$$E(r_A) \geq E(r_B)$$

- And

$$\sigma_A \leq \sigma_B$$

Q. How do you find a family of portfolios you are indifferent to?

Utility Indifference Curve



How Do You Estimate Risk Aversion?

- Use questionnaires
- Observe individuals' decisions when confronted with risk
- Observe how much people are willing to pay to avoid risk
- Use common sense

Capital Allocation Across Risky and Risk-Free Portfolios

Asset Allocation:

- Is a very important part of portfolio construction.
- Refers to the choice among broad asset classes.

Controlling Risk:

- Simplest way: Manipulate the fraction of the portfolio invested in risk-free assets versus the portion invested in the risky assets

Basic Asset Allocation

Portfolio Total Market Value	\$300,000
Risk-free money market fund	\$90,000
Equities	\$113,400
Bonds (long-term)	\$96,600
Total risk assets	\$210,000

$$W_E = \frac{\$113{,}400}{\$210{,}000} = 0.54 \qquad W_B = \frac{\$96{,}600}{\$210{,}00} = 0.46$$

These weights are within the *risky* portfolio

Q. What is the risk-free vs risky composition?

Basic Asset Allocation

- Let *y* = weight of the risky portfolio, P, in the complete portfolio; (1-*y*) = weight of risk-free assets:

$$y = \frac{\$210{,}000}{\$300{,}000} = 0.7 \qquad 1 - y = \frac{\$90{,}000}{\$300{,}000} = 0.3$$

$$E: \frac{\$113{,}400}{\$300{,}000} = .378 \qquad B: \frac{\$96{,}600}{\$300{,}000} = .322$$

These weights are within the *entire* portfolio

The Risk-Free Asset

- Only the government can issue default-free bonds (caveats).
  - Risk-free in real terms only if price indexed and maturity equal to investor’s holding period.
- T-bills viewed as “the” risk-free asset
- Money market funds also considered risk-free in practice
  (caveat, remember fall 2008?)

Figure 6.3 Spread Between 3-Month CD and T-bill Rates

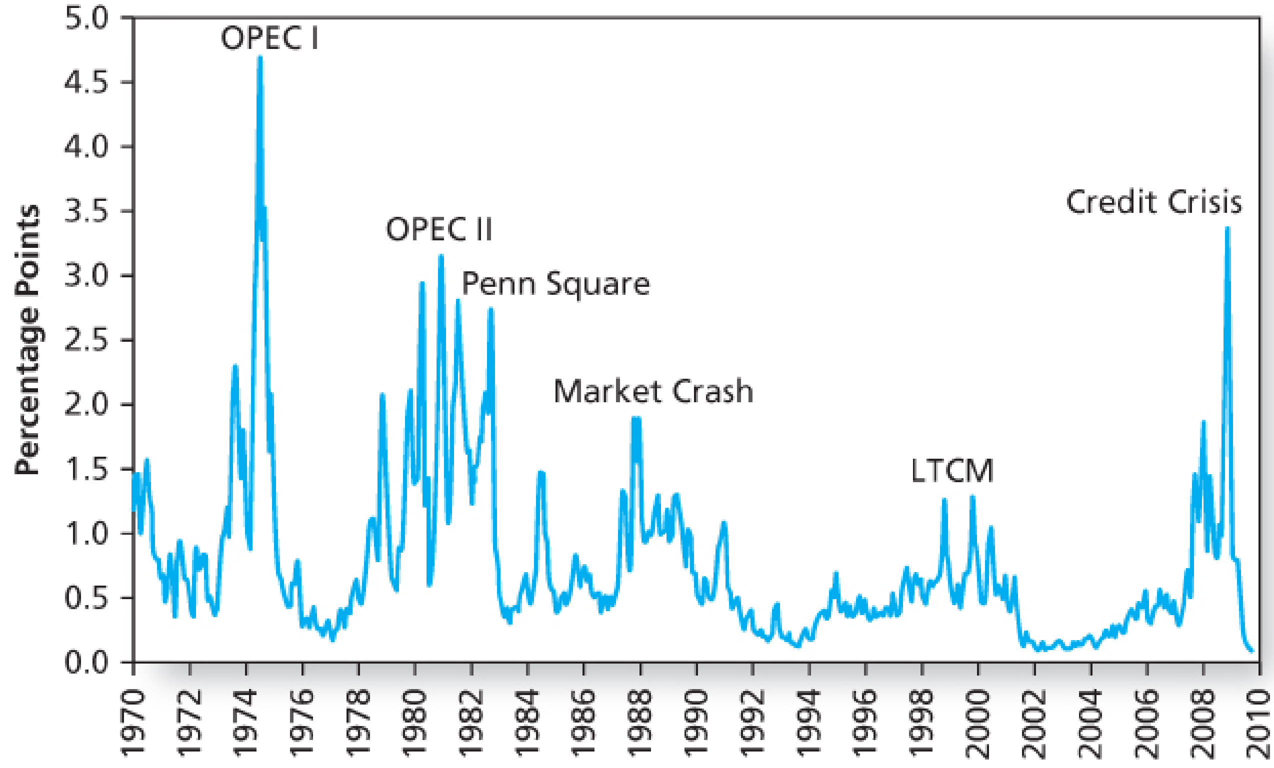


Figure 6.3 Spread between 3-month CD and T-bill rates

Portfolios of One Risky Asset and a Risk-Free Asset

You can create a complete portfolio by splitting funds between safe and risky assets. Let:

- y = portion allocated to the risky portfolio, P
- (1-y) = portion to invest in risk-free asset, F.

Build a complete portfolio C: $r_C = yr_p + (1 - y)r_f$

Take expectations: $E(r_C) = yE(r_p) + (1 - y)r_f$

Rearrange terms: $E(r_C) = r_f + yE\underbrace{(r_p - r_f)}_{\text{risk premium}}$

Q. What's the porfolio's σ_c?

Example Using Chapter 6.4 Numbers

Risky	Risk-free
$E(r_p) = 15\%$	$r_f = 7\%$
$\sigma_p = 22\%$	$\sigma_{rf} = 0\%$
y = % in p	$(1-y)$ = % in r_f

Example (Ctd.)

The expected return on the complete portfolio is the risk-free rate plus the weight of P times the risk premium of P

$$E(r_C) = r_f + yE\underbrace{(r_p - r_f)}_{\text{risk premium}}$$

$$E(r_c) = 7 + y(15 - 7)$$

Example (Ctd.)

- The risk of the complete portfolio is the weight of P times the risk of P because the risk free asset has zero standard deviation:

$$\sigma_C = y\sigma_P = 22y$$

Example (Ctd.)

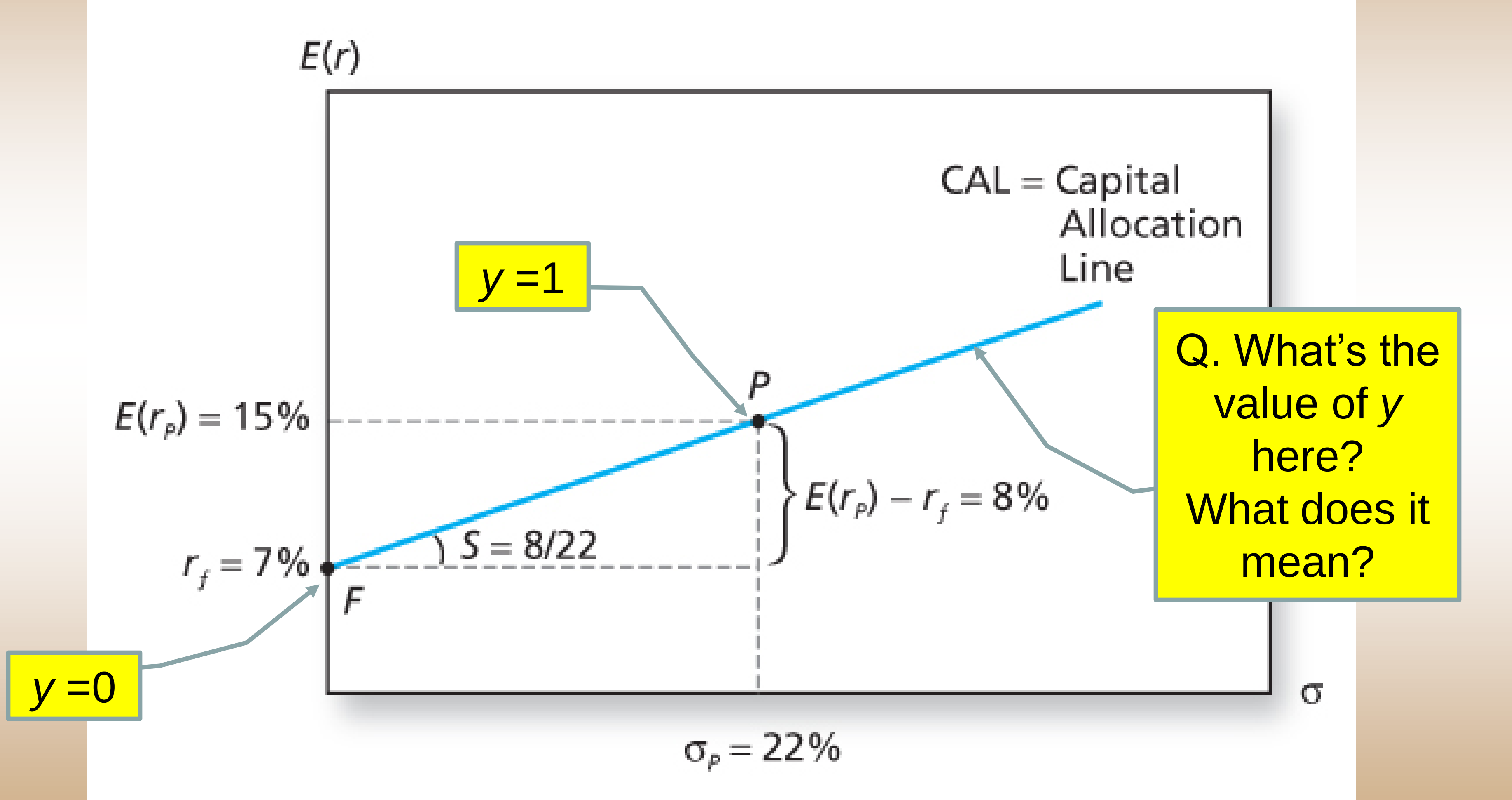
Place the two portfolios P and F on the {r,σ} plane. Varying *y* from 0 to 1 describes a line between F and P, what is the slope?

Rearrange and substitute $y=\sigma_C / \sigma_P$:

$$E(r_C) = r_f + \frac{\sigma_C}{\sigma_P}\left[E(r_P) - r_f\right] = 7 + \frac{8}{22}\sigma_C$$

$$Slope = \frac{E(r_P) - r_f}{\sigma_P} = \frac{8}{22} \qquad Intercept = r_f = 7$$

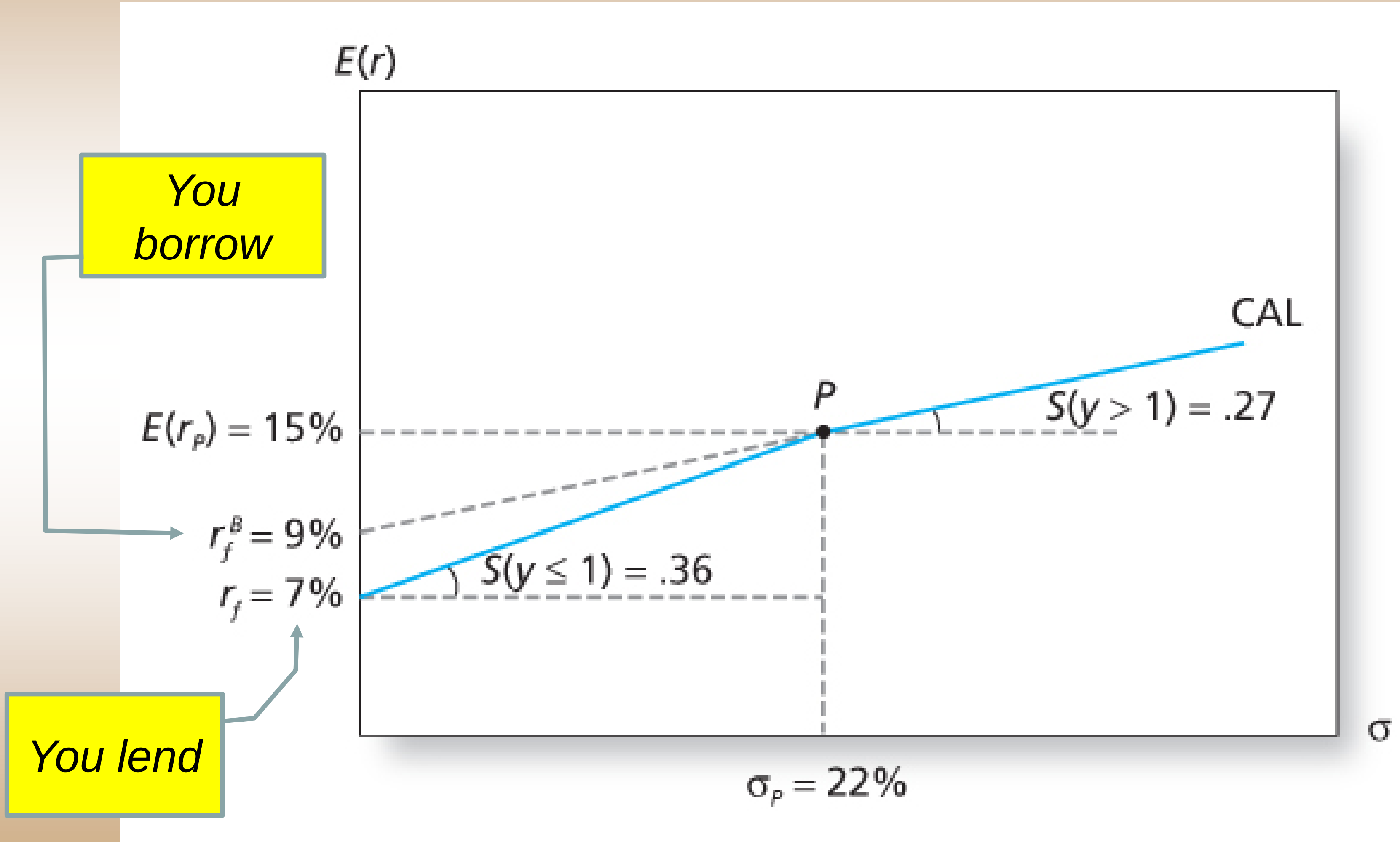
Figure 6.4 The Investment Opportunity Set



Capital Allocation Line with Leverage

- *y*>1 means borrow money to lever up your investment (e.g. buy on margin)
- There is asymmetry: lend (or invest) at r_f=7% and borrow at r_f=9%
  – Lending range slope = 8/22 = 0.36
  – Borrowing range slope = 6/22 = 0.27
- CAL kinks at P

Figure 6.5 The Opportunity Set with Differential Borrowing and Lending Rates



Risk Tolerance and Asset Allocation

- The investor must choose one optimal portfolio, *C*, from the set of feasible choices (by changing *y*)
  - Expected return of the complete portfolio:

$$E(r_C) = r_f + y\left(E(r_p) - r_f\right)$$

  - Variance:

$$\sigma_C^2 = y^2 \sigma_P^2$$

Utility Function depending on y

- Express U as a function of *y*

$$U = E(r_C) - \frac{1}{2} A\sigma_C^2$$

$$U = r_f + y\left[E(r_p) - r_f\right] - \frac{1}{2} A\left(y^2\sigma_P^2\right)$$

- U is a quadratic function of *y*

$$U = -ay^2 + yb + c$$

Table 6.4 Utility Levels for Various Positions in Risky Assets (y) for an Investor with Risk Aversion A = 4

(1) y	(2) $E(r_C)$	(3) σ_C	(4) $U = E(r) - \frac{1}{2}A\sigma^2$
0	.070	0	.0700
0.1	.078	.022	.0770
0.2	.086	.044	.0821
0.3	.094	.066	.0853
0.4	.102	.088	.0865
0.5	.110	.110	.0858
0.6	.118	.132	.0832
0.7	.126	.154	.0786
0.8	.134	.176	.0720
0.9	.142	.198	.0636
1.0	.150	.220	.0532

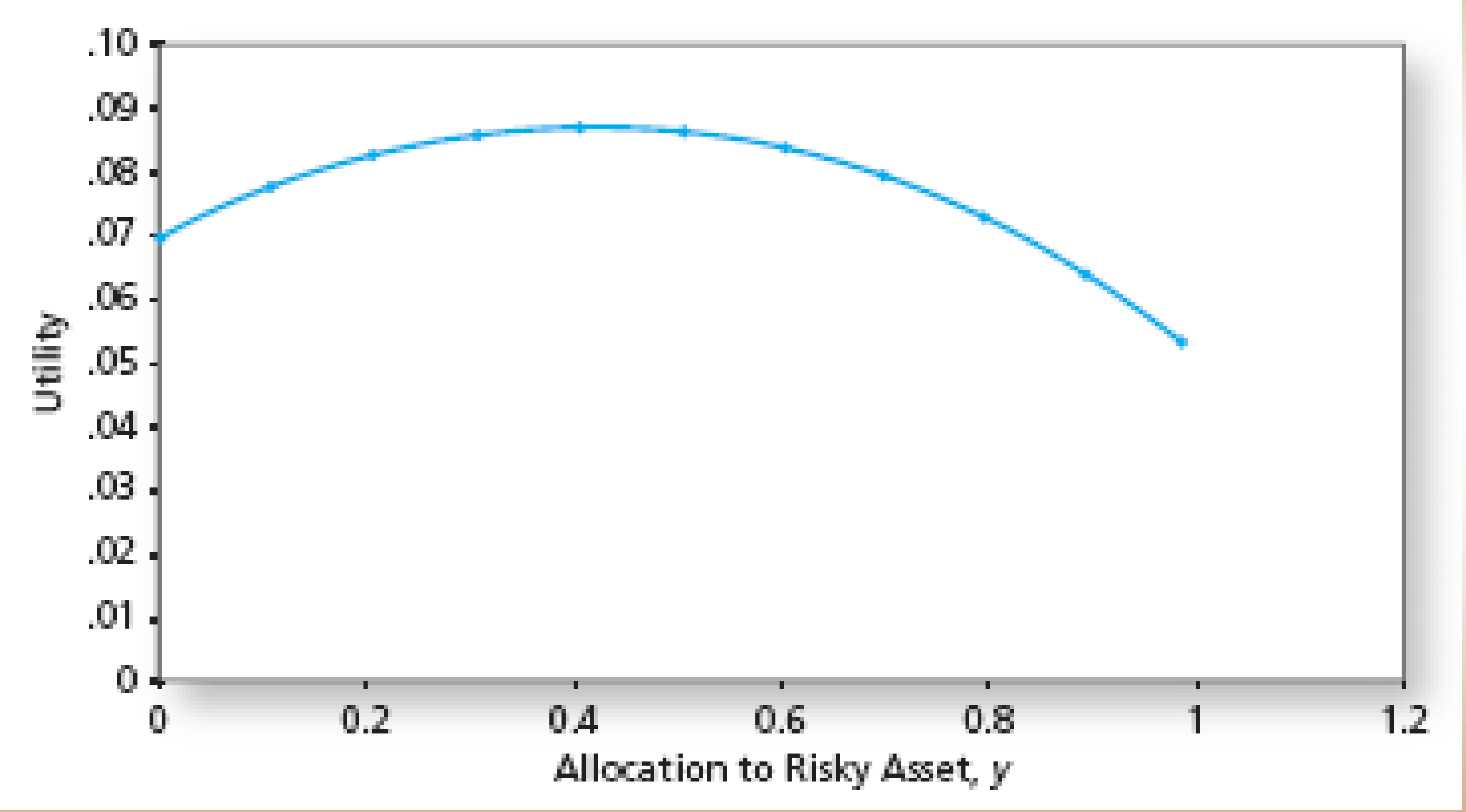
Example:

$r_f = 7\%$

$E(r_p) = 15\%$

$\sigma_p = 22\%$

Figure 6.6 Chart Utility as a Function of the Allocation to the Risky Asset (*y*)



Maximize Utility Function w.r.t. *y*

- Express U as a function of *y*

$$U = E(r_C) - \frac{1}{2}A\sigma_C^2$$

$$U = r_f + y\left[E(r_p) - r_f\right] - \frac{1}{2}A\left(y^2\sigma_P^2\right)$$

- The maximize w.r.t. *y*

$$y_{\max U} = \frac{E(r_p) - r_f}{A\sigma_P^2}$$

Utility Indifference curves

$$U = E[r] - \frac{1}{2}A\sigma^2 = const$$

For example:

$$U = 0.05, 0.09$$

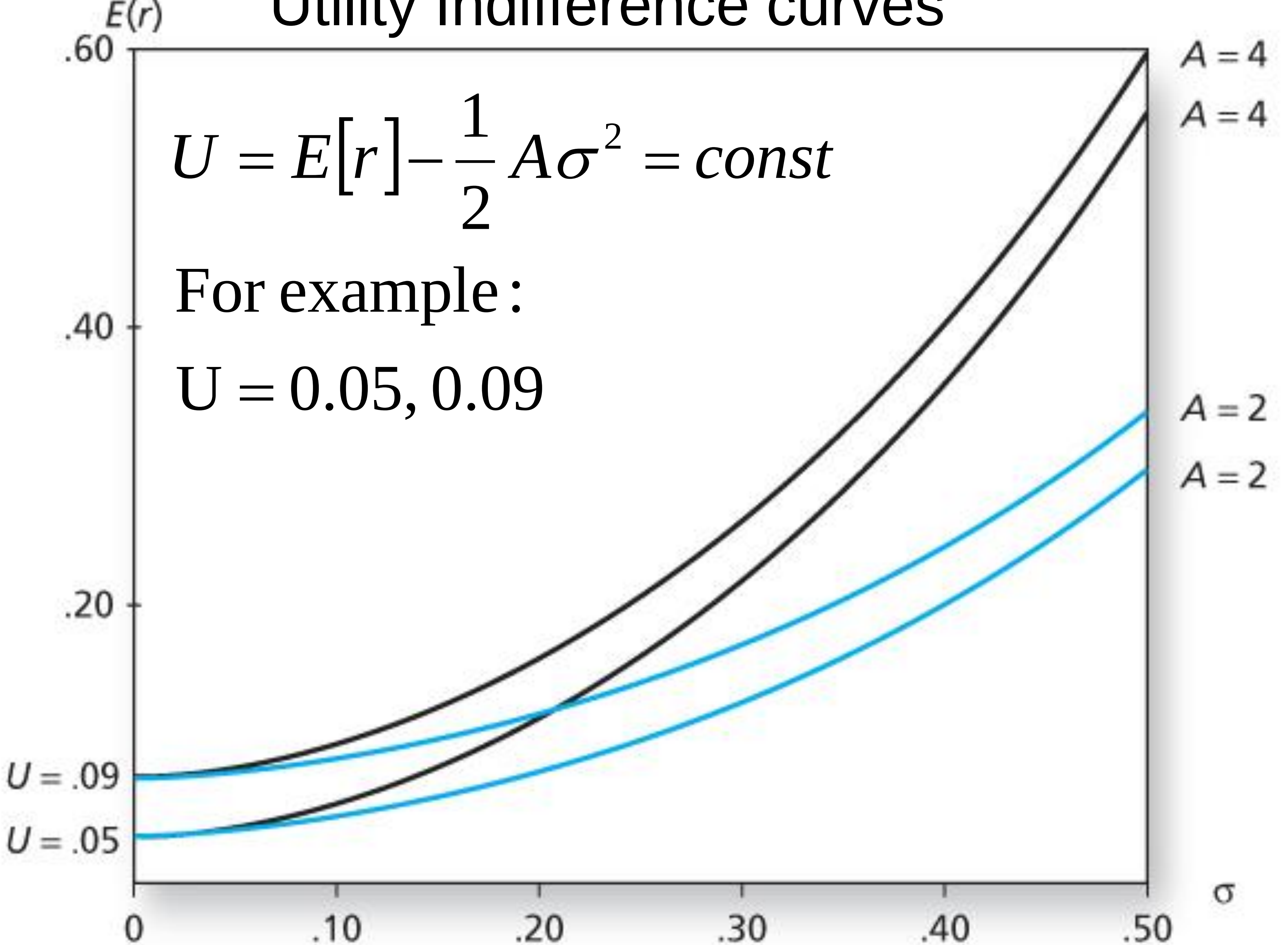


Table 6.5 Spreadsheet Calculations of Indifference Curves

Table 6.5

Spreadsheet calculations of indifference curves (Entries in columns 2–4 are expected returns necessary to provide specified utility value.)

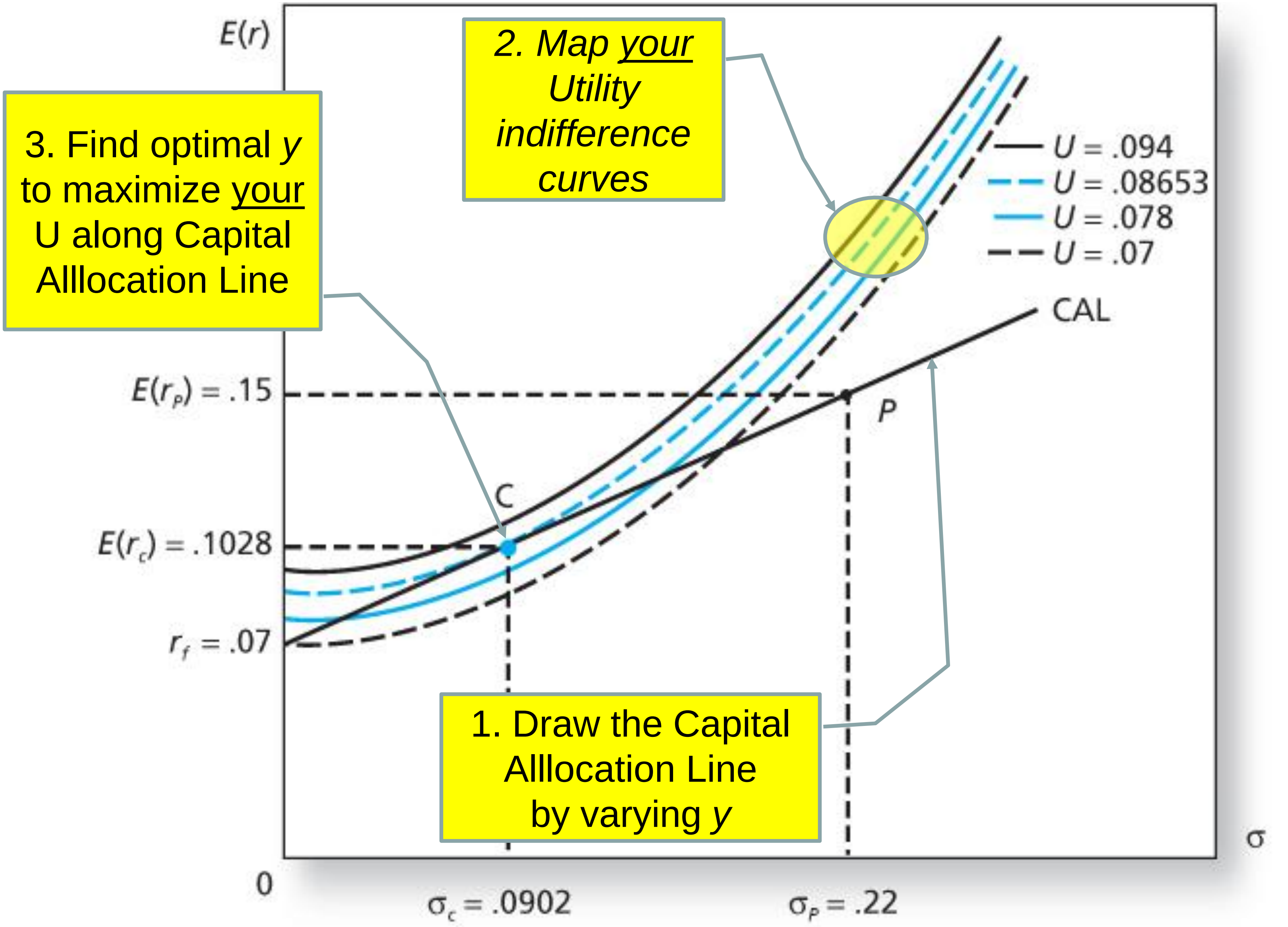
	A = 2		A = 4	
σ	U = .05	U = .09	U = .05	U = .09
0	.0500	.0900	.050	.090
.05	.0525	.0925	.055	.095
.10	.0600	.1000	.070	.110
.15	.0725	.1125	.095	.135
.20	.0900	.1300	.130	.170
.25	.1125	.1525	.175	.215
.30	.1400	.1800	.230	.270
.35	.1725	.2125	.295	.335
.40	.2100	.2500	.370	.410
.45	.2525	.2925	.455	.495
.50	.3000	.3400	.550	.590

Table 6.6 Expected Returns on Four Indifference Curves and the CAL

Risk aversion coefficient A=4

σ	U = .07	U = .078	U = .08653	U = .094	CAL
0	.0700	.0780	.0865	.0940	.0700
.02	.0708	.0788	.0873	.0948	.0773
.04	.0732	.0812	.0897	.0972	.0845
.06	.0772	.0852	.0937	.1012	.0918
.08	.0828	.0908	.0993	.1068	.0991
.0902	.0863	.0943	.1028	.1103	.1028
.10	.0900	.0980	.1065	.1140	.1064
.12	.0988	.1068	.1153	.1228	.1136
.14	.1092	.1172	.1257	.1332	.1209
.18	.1348	.1428	.1513	.1588	.1355
.22	.1668	.1748	.1833	.1908	.1500
.26	.2052	.2132	.2217	.2292	.1645
.30	.2500	.2580	.2665	.2740	.1791

Put it together and find your optimal allocation



Passive Strategies: The Capital Market Line

- The passive strategy avoids any direct or indirect security analysis
- Supply and demand forces may make such a strategy a reasonable choice for many investors

Passive Strategies: The Capital Market Line

- A natural candidate for a passively held risky asset would be a well-diversified portfolio of common stocks such as the S&P 500.
- The capital market line (CML) is the capital allocation line formed from 1-month T-bills and a broad index of common stocks (e.g. the S&P 500).

Passive Strategies: The Capital Market Line

- The CML is given by a strategy that involves investment in two passive portfolios:

  1. a *virtually* risk-free portfolio of short-term T-bills (or a money market fund)
  2. a fund of common stocks that mimics a broad market index.

Passive Strategies: The Capital Market Line

- From 1926 to 2009, the passive risky portfolio offered an average risk premium of 7.9% with a standard deviation of 20.8%, resulting in a reward-to-volatility ratio of .38.