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AN EFFECTIVE APPROACH FOR SOLVING THE BINARY ASSIGNMENT PROBLEM WITH SIDE CONSTRAINTS

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The binary assignment problem with a side constraint requiring the objective function to receive a specified value, which in general is an NP-hard problem, has been the focus of several papers in recent years. The current literature addresses various theoretical aspects of the problem, with a particular emphasis on a simplifying special case, but stops short of giving any computational experience. In this paper, we present a simple reformulation that enables the problem, and some of its extensions, to be solved by commonly available heuristic methods. We present preliminary computational experience with a Tabu search method that illustrates the effectiveness and computational robustness of the approach.

Keywords: Assignment problems; optimization; integer programming; heuristics; reformulation.

1. Introduction

The side constrained binary assignment problem (SC-BAP, or BAP for short), may be formulated as follows:

BAP: Find a feasible binary solution, x_{ij}, i, $j \in N = \{1, \ldots, n\}$ to the equations

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i \in N \tag{1}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j \in N \tag{2}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} = r \tag{3}$$

where c_{ij} and r are integers unrestricted in sign. Equation (3) is called the side constraint.

Applications of BAP associated with core management of pressurized water nuclear reactors have been reported.[5,9] Other applications,[19] include analysis of medical images by Tanimoto (1976) and timetabling by Even, *et al.* (1976).

The general BAP problem[6] is shown to be NP-hard. For the special case where the matrix $C = (c_{ij} : i, j \in N)$ is binary and the graph is complete, polynomial algorithms for BAP are given.[14,19] Analysis of the polyhedral characterization of this special case is given[1] while Ref. 2, again for this special case, describes facet-inducing inequalities. None of these papers, even for the indicated special case, report any computational experience.

In this paper, we propose a solution procedure that starts with a simple reformulation enabling the problem to be solved by readily available heuristics. Preliminary computational experience is given to illustrate the approach and to document its relevance and promise for solving significant instances of BAP.

2. Reformulation

Our approach is based on introducing a quadratic infeasibility penalty function to implicitly enforce constraints (in this case the side constraint) rather than explicitly representing such restrictions in the form of traditional constraints. Variants of this type of transformation have been applied in other problem contexts.[10,11,16] Examples illustrating this type of reformulation approach to create an unconstrained binary quadratic programming representation that models a wide variety of zero-one problems is given in Ref. 12.

For BAP, we perform the transformation in a way that causes the resulting optimization problem to take the form of a quadratic assignment problem (QAP), which can then be solved by any of a variety of available solution methods. While the QAP is itself a difficult (NP-Hard) problem, several heuristic methods have recently proven to perform well on large-scale instances. Moreover, in the present case, knowledge about the optimum objective function value to QAP, if a feasible BAP solution exists, can be used to permit early termination of the search.

The objective function for our transformed problem consists of minimizing

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - r\right)^2 \tag{4}$$

To simplify the notation in what follows, we re-label both parameters and variables using single subscripts in the natural way from 1 to $m = n^2$

$$(x_1 \ldots x_n, \ldots, \ldots x_m) = (x_{11} \ldots x_{1n}, \ldots, \ldots x_{nm})$$

and similarly reindex the c_{ij} coefficients to receive single subscripts. Then, the objective function (4) can be re-written in the form

$$xQx + r^2$$

where

$$q_{ii} = c_i^2 - 2rc_i$$
$$q_{ij} = c_i c_j \quad \text{for} \quad i \neq j$$

Dropping the additive constant (r^2), we have the re-formulated equivalent of BAP that takes the quadratic assignment form:

QAP: Minimize $x_0 = xQx$ subject to equations (with re-labeled variables) (1) and (2).

A solution to QAP *that solves* BAP must have $x_0 = -r^2$. Thus, a heuristic used to solve QAP can be terminated whenever $x_0 = -r^2$; that is, $-r^2$ becomes a *target* the heuristic procedure attempts to achieve.

Example: This approach is illustrated by the following 3×3 example.

BAP:

$$\begin{array}{lllllllllll}
4x_1 & +6x_2 & +8x_3 & +6x_4 & +6x_5 & +4x_6 & +8x_7 & +4x_8 & +6x_9 & = 18 \\
x_1 & +x_2 & +x_3 & & & & & & & = 1 \\
& & & x_4 & +x_5 & +x_6 & & & & = 1 \\
& & & & & & x_7 & +x_8 & +x_9 & = 1 \\
x_1 & & & +x_4 & & & +x_7 & & & = 1 \\
& x_2 & & & +x_5 & & & +x_8 & & = 1 \\
& & x_3 & & & +x_6 & & & +x_9 & = 1
\end{array}$$

Transforming the side constraint as indicated above gives the equivalent QAP problem:

QAP: Minimize $x_0 = xQx$ subject to

$$\begin{array}{llllllllll}
x_1 & +x_2 & +x_3 & & & & & & & = 1 \\
& & & x_4 & +x_5 & +x_6 & & & & = 1 \\
& & & & & & x_7 & +x_8 & +x_9 & = 1 \\
x_1 & & & +x_4 & & & +x_7 & & & = 1 \\
& x_2 & & & +x_5 & & & +x_8 & & = 1 \\
& & x_3 & & & +x_6 & & & +x_9 & = 1
\end{array}$$

where the optimal (target) value for x_0 is given by $-r^2 = -324$ and the Q matrix is

$$Q = \begin{bmatrix} -128 & 24 & 32 & 24 & 24 & 16 & 32 & 16 & 24 \\ 24 & -180 & 48 & 36 & 36 & 24 & 48 & 24 & 36 \\ 32 & 48 & -224 & 48 & 48 & 32 & 64 & 32 & 48 \\ 24 & 36 & 48 & -180 & 36 & 24 & 48 & 24 & 36 \\ 24 & 36 & 48 & 36 & -180 & 24 & 48 & 24 & 36 \\ 16 & 24 & 32 & 24 & 24 & -128 & 32 & 16 & 24 \\ 32 & 48 & 64 & 48 & 48 & 32 & -224 & 32 & 48 \\ 16 & 24 & 32 & 24 & 24 & 16 & 32 & -128 & 24 \\ 24 & 36 & 48 & 36 & 36 & 24 & 48 & 24 & -180 \end{bmatrix}$$

Solving this quadratic assignment problem yields $x_2 = x_6 = x_7 = 1$ for which $x_0 = -324$. Since x_0 achieves the target level, this solution is feasible for BAP.

3. Computational Experience

The equivalent QAP models representing BAP can be solved by any of the several new approaches to QAP problems. The reader is referred to recent articles[3,4,7,13,15,17,18] for descriptions of representative methods. For the work reported here, we use a straightforward Tabu search method that employs short and long term memory to guide the search process. Details of the Tabu search components used here are given in Ref. 8.

In this section, we present our experience with problem sizes of 100, 225, and 400 variables with three instances considered for each problem size. The c_j values were randomly chosen from the uniform interval (1–10). For each problem, the right hand side value, r, was chosen so that at least one feasible solution exists.

As indicated in Table 1, our approach was able to quickly find optimal QAP solutions (hence feasible BAP solutions) to all of the test problems in the test bed. The largest of the problems (those with 400 variables) were solved in 3 seconds or less on a Pentium 200 PC. Based on our experience with problems of similar structure but larger size, we would expect to solve instances of BAP with up to 1500 variables in a few minutes with our Tabu search method.

Table 1. Computational experience.

ID	m	r^2	x_0	Time
$A1$	100	3136	3136	< 1 sec
$A2$	100	1849	1849	< 1 sec
$A3$	100	2916	2916	< 1 sec
$B1$	225	6561	6561	< 1 sec
$B2$	225	2304	2304	< 1 sec
$B3$	225	6889	6889	< 1 sec
$C1$	400	2401	2401	2 sec
$C2$	400	4489	4489	3 sec
$C3$	400	7396	7396	2 sec

4. Extensions of the Basic Model

Several extensions of the base model can be readily accommodated by the approach of this paper. In this section, we highlight three such extensions.

4.1. *Missing variables*

Reference 2 describes the case where one or more variables are missing from the problem or where, for problem specific reasons, certain variables are "forbidden" to be chosen. The polynomial algorithm they give to solve the special case where the C matrix is binary will not work on problems with missing variables. In fact, they comment (p. 366) that whether an efficient algorithm exists for such problems "... remains an open question".

Our approach to solving BAP, for both the binary and general coefficient cases, is unaffected by missing or forbidden variables. Each such variable is readily forced to be zero by placing a large penalty in each of the row and column entries of the Q matrix that correspond to the variable in question. (An alternative would be to simply remove each missing variable from the problem. This, however, would alter the QAP structure that we are utilizing in our approach here and thus we have adopted the penalty approach.)

As an illustration, consider again the example of Sec. 2 where a feasible solution was obtained with x_2, x_6 and x_7 equal to 1. Suppose we now want to consider x_6 to be a forbidden variable, implying that a solution is required with $x_6 = 0$. This can readily be accomplished by choosing a large positive penalty P and solving QAP with the penalized Q matrix

$$Q = \begin{bmatrix} -128 & 24 & 32 & 24 & 24 & P & 32 & 16 & 24 \\ 24 & -180 & 48 & 36 & 36 & P & 48 & 24 & 36 \\ 32 & 48 & -224 & 48 & 48 & P & 64 & 32 & 48 \\ 24 & 36 & 48 & -180 & 36 & P & 48 & 24 & 36 \\ 24 & 36 & 48 & 36 & -180 & P & 48 & 24 & 36 \\ P & P & P & P & P & P & P & P & P \\ 32 & 48 & 64 & 48 & 48 & P & -224 & 32 & 48 \\ 16 & 24 & 32 & 24 & 24 & P & 32 & -128 & 24 \\ 24 & 36 & 48 & 36 & 36 & P & 48 & 24 & -180 \end{bmatrix}.$$

Solving this new QAP example, with P arbitrarily chosen to be 600, gives the result $x_2 = x_4 = x_9 = 1$ for which $x_0 = -324$ and, as desired, our forbidden variable is equal to zero. Since we have again reached the target level for x_0, this solution is feasible for BAP.

4.2. *Multiple side constraints*

Our approach for solving BAP can also readily accommodate multiple side constraints. Each such side constraint, in equation form, can be converted into a quadratic penalty function exactly the way (3) was converted, and the Q matrices produced can be simply added together to get a "grand" Q matrix for the equivalent QAP problem. To illustrate, return to the example of Sec. 2 and consider a *second* side constraint given by:

$$3x_1 + 3x_2 + 5x_3 + 6x_4 + 7x_5 + 4x_6 + 5x_7 + 3x_8 + 4x_9 = 13$$

Converting this constraint to a quadratic penalty function and combining the associated Q matrix with that of the first side constraint yields the QAP with a (combined) Q matrix given by

$$Q = \begin{bmatrix} -197 & 33 & 47 & 42 & 45 & 28 & 47 & 25 & 36 \\ 33 & -249 & 63 & 54 & 57 & 36 & 63 & 33 & 48 \\ 47 & 63 & -329 & 78 & 83 & 52 & 89 & 47 & 68 \\ 42 & 54 & 78 & -300 & 78 & 48 & 78 & 42 & 60 \\ 45 & 57 & 83 & 78 & -313 & 52 & 83 & 45 & 64 \\ 28 & 36 & 52 & 48 & 52 & -216 & 52 & 28 & 40 \\ 47 & 63 & 89 & 78 & 83 & 52 & -329 & 47 & 68 \\ 25 & 33 & 47 & 42 & 45 & 28 & 47 & -197 & 36 \\ 36 & 48 & 68 & 60 & 64 & 40 & 68 & 36 & -269 \end{bmatrix}$$

Note that the target level for this expanded problem is or $-(r_1^2 + r_2^2)$ or -493. Solving QAP by our Tabu search method gives $x_2 = x_4 = x_9 = 1$ for which x_0 is -493. Since our target level has been reached, this solution satisfies both of the side constraints as well as the assignment constraints.

4.3. *Explicit objective function*

The previous cases started as satisfiability problems that were subsequently converted into optimization problems via the introduction of an objective function derived from a quadratic penalty function. A generalization is to consider the problem of finding a bipartite matching that simultaneously satisfies a side constraint while minimizing an explicit linear objective function.

This more general problem can be stated by

$$\min x_0 = \sum_{j=1}^{m} a_j x_j$$

subject to (1)–(3).

(As earlier, we reindex double subscripted variables and parameters to put them in a single subscript form.) A special case of this generalization was mentioned by Yi (1994) as a topic for further research to extend the work of his dissertation. However, to the present date, no work addresses this problem.

The model that includes an explicit objective function, like the previous extensions of the base model, can be reformulated as:

QAP: min xQx subject to (1) and (2)

where

$$q_{ii} = a_i + M(c_i^2 - 2rc_i)$$

$$q_{ij} = Mc_ic_j \quad \text{for} \quad i \neq j$$

The constant M is a positive scalar penalty chosen large enough to ensure that the side constraint, through the construction of the quadratic penalty function, is satisfied. A suitable choice for M can always be made.[10] For the problems considered here, assuming all data are integers, any value larger that the sum of the absolute values of the a_j coefficients will work. Note that we can readily carry this reformulation a step further by supposing that the original objective function is not linear, but is itself a quadratic function. The rules for handling this are analogous to those previously described.

To illustrate the procedure, consider modifying our original example by including the objective function

$$\min\, x_0 = 5x_1 + 2x_2 + 2x_3 + 6x_4 + 9x_5 + 9x_6 + 4x_7 + 5x_8 + 3x_9$$

which is to be optimized subject to the side constraint and assignment constraints shown in Sec. 2. Choosing the penalty M to be 50, and recasting this problem into the QAP format by our indicated transformation gives the equivalent QAP problem with the Q matrix:

$$Q = \begin{bmatrix} -6395 & 1200 & 1600 & 1200 & 1200 & 800 & 1600 & 800 & 1200 \\ 1200 & -8996 & 2400 & 1800 & 1800 & 1200 & 2400 & 1200 & 1800 \\ 1600 & 2400 & -11198 & 2400 & 2400 & 1600 & 3200 & 1600 & 2400 \\ 1200 & 1800 & 2400 & -8994 & 1800 & 1200 & 2400 & 1200 & 1800 \\ 1200 & 1800 & 2400 & 1800 & -8991 & 1200 & 2400 & 1200 & 1800 \\ 800 & 1200 & 1600 & 1200 & 1200 & -6391 & 1600 & 800 & 1200 \\ 1600 & 2400 & 3200 & 2400 & 2400 & 1600 & -11196 & 1600 & 2400 \\ 800 & 1200 & 1600 & 1200 & 1200 & 800 & 1600 & -6395 & 1200 \\ 1200 & 1800 & 2400 & 1800 & 1800 & 1200 & 2400 & 1200 & -8997 \end{bmatrix}$$

Solving this quadratic assignment problem yields the solution: $x_2 = x_4 = x_9 = 1$ for which the original objective function is equal to 13. It is easy to check that this solution satisfies all problem constraints and in fact is optimal.

5. Summary and Conclusions

We have presented a reformulation of the assignment problem with side constraints that enables solutions to be computed using commonly available heuristic procedures for the quadratic assignment problem. Our approach is not restricted to special cases, nor does it rely on simplifying assumptions concerning the completeness of the bipartite matching sought. To illustrate the robustness of our approach, we have shown how various extensions of the base model, including missing variables, multiple side constraints, and explicit objective functions, can be readily modeled and solved by the same reformulation.

Computational experience using a straightforward Tabu search method for problems up to size 400 discloses the practical viability of the procedure. Our computational experience, the first to be offered for this class of problems, suggests that the approach outlined here is both very effective and efficient. Specialized solution methods for larger problems and further extensions of the base model are the subject of on-going work.

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References

1. A. Y. Alfakih, Facets of an assignment problem with a 0-1 side constraint, PhD Dissertation, University of Michigan, Ann Arbor, 1996.
2. A. Y. Alfakih, T. Yi and K. G. Murty, Facets of an assignment problem with 0-1 side constraint, *J. Combinatorial Optimization* **4** (2000) 365–388.
3. R. Battiti and G. Tecchiolli, The reactive Tabu search, *ORSA J. Comput.* **6** (1994) 126–140.
4. R. E. Burkard, S. E. Karish and F. Rendl, QAPLIB — A Quadratic Assignment Problem Library, www.opt.math.turgraz.at/~karish/qaplib (1994).
5. J. Brans, M. Leclercq and P. Hansen, An algorithm for optimal reloading of pressurized water reactors, *Operations Res.*, ed. M. Ross (North Holland, 1973) 417–428.
6. R. Chandrasedaran, S. N. Kabadi and K. G. Murty, Some NP-hard problems in linear programming, *OR Lett.* 1 (1982) 101–104.
7. G. Fleurent and F. Glover, Improved constructive multistart strategies for the quadratic assignment problem using adaptive memory, *Inf. J. Comput.* **11**, 2 (spring 1999) 198–204.
8. F. Glover, G. Kochenberger, B. Alidaee and M. Amini, Tabu search with critical event memory: An enhanced application for binary quadratic programs, *Meta Heuristics: Advances and Trends in Local Search Paradigms for Optimization*, eds. S. Voss, *et al.* (Kluwer Publishers, 1999) Chapter 7.
9. A. Gupta and J. Sharma, Tree search method for optimal core management of pressurized water reactors, *Comput. Operations Res.* **8** (1981) 263–266.
10. P. Hammer and S. Rudeanu, *Boolean Methods in Operations Research* (Springer-Verlag, New York, 1968).
11. P. Hansen, Methods of nonlinear 0-1 programming, *Ann. Discrete Math.* **5** (1979) 53–70.
12. G. Kochenberger, B. Alidaee and M. Amini, Applications of the unconstrained quadratic binary program, Working Paper, University of Colorado, 1998.
13. Y. Li, P. Pardalos and M. Resende, A greedy randomized adaptive search procedure for the quadratic assignment problem, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **16** (1994) 237–261.

14. K. G. Murty, C. Spera and T. Yi, Matchings in colored networks, Working Paper, IOE Department, University of Michigan, 1993, revised 2000.
15. P. Pardalos, F. Rendl and H. Wolkowicz, The quadratic assignment problem: A survey and recent developments, *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **16** (1994) 1–42.
16. J. C. Picard and D. Ratliff, A graph-theoretic equivalence for integer programs, *OR* **21**, 1 (1973) 261–269.
17. H. Sherali and P. Rajgopal, A flexible, polynomial-time, constructive and improvement heuristic for the quadratic assignment problem, *Comput. Operations Res.* **13**, 5 (1986) 587–600.
18. E. D. Taillard, Comparison of iterative searches for the quadratic assignment problem, *Publication CRT-989*, Centre de rescherche sur les transport, Montreal, 1989.
19. T. Yi, Bipartite matching with specified values for a 0-1 linear function, PhD Dissertation, The University of Michigan, Ann Arbor, 1994.