A NOTE ON LINEAR PROGRAMMING AND INTEGER FEASIBILITY

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This paper proves a theorem that provides new strategies for solving integer programming problems, based on finding certain types of basic solutions to linear programs. The theorem is motivated by and extends ideas of CABOT AND HURTER. An integer programming method based on the theorem is outlined.

THE zero-one mixed integer programming problem may be written

$$\text{Maximize } cx+dy, \text{ subject to } Ax+Dy\leqq b, \quad x\leqq e,\ x,\ y\geqq 0 \text{ and } x \text{ integer}, \tag{1}$$

where e denotes a vector of ones, A is $m\times n$, D is $m\times r$, x is $n\times 1$, y is $m\times 1$, and c, d, b, and e are dimensioned compatibly. Adding vectors u and v of slack variables, the constraints of (1) become

$$Ax+Dy+u=b,\ x+v=e,\ x,\ y,\ u,\ v\geqq 0 \text{ and } x \text{ integer}. \tag{2}$$

The results to follow are unchanged if this constraint is more generally $Ax+Dy+Uu=b$, provided the augmented matrix (D, U) contains an $m\times m$ nonsingular submatrix.

A strategy that is often used for solving (1) is to adjoin additional constraints and variables to partition the feasible region into subsets (e.g., restricting $cx+dy$ to specified intervals), and to seek a feasible solution to the additionally constrained problem. We assume that such constraints and variables are already included in (1) and (2), and address ourselves to obtaining feasible solutions to (2).

In a variety of practical situations the constraints of (2) contain imbedded network problems (such as transportation and assignment problems) having the property that every extreme-point solution is integer. Moreover, one can readily derive constraints from (2) that impose bounds on nested partial sums of variables (*see* reference 3), and these constraints also have the property that every extreme point is integer. Finally, a set of such constraints can be adjoined to further partition the feasible region of (2).

Thus, representing these special constraints by $Px=f$ and $Qx+w=g$, consider the augmented system

$$Ax+Dy+u=b,\ x+v=e,\ Px=f,\ Qx+w=g,\qquad x,\ y,\ u,\ v,\ w\geqq 0 \text{ and } x \text{ integer}, \tag{3}$$

where P, Q, f, and g are integer matrices, P is $p\times n$, Q is $q\times n$, and $\begin{pmatrix}P\\Q\end{pmatrix}$ has the unimodular property; i.e., every square submatrix of $\begin{pmatrix}P\\Q\end{pmatrix}$ has determinant 0, 1, or -1. (P or Q may also be null.) We also stipulate $p\leqq n$ and every $p\times p$ submatrix of P is nonsingular.

THEOREM. *If there is a feasible solution to* (3) *with* $x=x'$ *and* x' *integer, then there is a basic feasible solution to* (3) *with* $x=x'$ *and* m *of the components of* (y, u) *basic. Moreover, every basic feasible solution to* (3) *with* m *of the components of* (y, u) *basic assigns integer values to the components of* x, v, *and* w.

The chief significance of this theorem is that it permits one to elect a strategy for solving (2) that focuses on finding a solution to (3) with m of the components of (y, u) basic.

The first application of such a strategy occurs in the pure zero-one linear programming method of CABOT AND HURTER,[1] whose ideas motivate this note. Specifically, the Cabot and Hurter method results (for D and d null) from adjoining the constraint $ex=N$ to (2) and replacing b by $b+\epsilon e$, where $0<\epsilon<1$ and A and b are assumed integer. Beginning with N at an upper bound for ex, Cabot and Hurter prescribe finding a basic feasible solution to this particular version of (3) with all components of u basic. (No procedure is given for accomplishing this, however.) If an acceptable solution is found, the method stops [or is applied to a new system (2)]. Otherwise, N is decremented and the process repeats.

The theorem implies that $ex=N$ can be dispensed with in the Cabot and Hurter approach, making it unnecessary to reapply the process for different values of N. Moreover, a variety of other side conditions and supplementary constraints can be accommodated by the theorem, some of the more important of which have already been indicated.

A method for exploiting the theorem for the pure integer programming problem is given below. We first establish the validity of the theorem with the following three lemmas.

LEMMA 1. *Every square submatrix of*

$$H=\begin{pmatrix} I & I & 0 \\ P & 0 & 0 \\ Q & 0 & I \end{pmatrix}$$

has determinant 0, 1, *or* -1 (*where the I matrices of the top row are* $n\times n$).

Proof. First, it is assumed that $\begin{pmatrix} P \\ Q \end{pmatrix}$ has the unimodular property. Using induction on q, and expanding the determinants of the appropriate submatrices of H by minors, it is easy to see that $\begin{pmatrix} P0 \\ QI \end{pmatrix}$ has the unimodular property. The rest of the proof follows the same argument, using induction on n.

LEMMA 2. *Consider the system*

$$Mt+Rz=c,\ Hz=\beta, \quad \text{and } t, \quad z\geqq 0, \tag{4}$$

where M *is* $m\times l$, R *is* $m\times s$, *and* H *is* $h\times s$, *the vectors* α *and* β *dimensioned compatibly. If* β *is integer and every square submatrix of* H *has determinant* 0, 1 or -1, *then* z *is integer in every basic solution of* (4) *with* m *of the components of* t *basic.*

Proof. The basic solution must have the form $t=M_1^{-1}(c-R_1H_1^{-1}\beta)$ and $Z=H_1^{-1}\beta$, where M_1 is an $m\times m$ submatrix of M, R_1 is an $m\times h$ submatrix of R, and H_1 is an $h\times h$ submatrix of H. The latter assures z is integer (*see* HOFFMAN AND KRUSKAL[5]).

Lemmas 1 and 2 collectively imply the latter part of the theorem. The first part of the theorem is implied by the following stronger statement.

LEMMA 3. *If there is a feasible solution to* (3) *with* $x=x'$ *and* x' *integer, then there is a basic feasible solution to* (3) *in which*

(i) *every component of* w *is basic,*

(ii) x_j *is basic if* $x_j'=1$ *and* v_j *is basic if* $x_j'=0$,

(iii) *any* p *of the remaining components of* (x, v) *are basic,*

(iv) m *of the components of* (y, u) *are basic.*

Proof. Conditions (i), (ii), (iii) give $q+n+p$ variables. We show that the submatrix composed of the associated columns of H of Lemma 1 (call it H_1) is nonsingular. First, all columns of Q in H may be reduced to 0 by subtracting from them appropriate multiples of the columns of $I_{q\times q}$ (associated with w). Next, there must be exactly p indices j such that x_j and v_j are chosen to be basic. Subtracting each of these v_j columns from its associated x_j column and rearranging rows and columns transform H_1 into $\begin{pmatrix} P_1 & R \\ 0 & I \end{pmatrix}$, where P_1 is a $p\times p$ submatrix of P, R consists of 0 columns and columns of P, and I is the $(n+q)\times(n+q)$ identity matrix. The nonsingularity of P_1 assures H_1 is nonsingular, and hence a basis for the subsystem of (3) with $Ax+Dy+u=b$ removed. Since H_1 contains a column for each positive component of x', t', and w' (where $t'=e-x'$, $w'=g-Qx'$), the basic solution to the subsystem must yield $x=x'$. Finally, (iv) is established in conjunction with (i), (ii), and (iii) from the assumed existence of a feasible, and hence a basic feasible, solution to $Dy+u=b-Ax'$.

AN INTEGER PROGRAMMING METHOD

WE GIVE an integer programming method for the pure zero-one problem (with D and d null) that pursues the objective of making the m components of u basic in (3). Assume A and b are integer, and replace A by $2A$ and b by $2b+e$. (This replacement clearly does not change the set of integers x satisfying $Ax\leqq b$, and implies $u\geqq e$ and integer for all nonnegative u, x satisfying $u+Ax=b$ and x integer.)

1. Solve the linear program: Maximize $x_0=au$, subject to (3) (disregarding the integer restriction on x), where $a>0$ and integer (e.g., let $a=e$). Represent the current tableau for the simplex method in the form

$$\text{Maximize } x_0=a_{00}+\textstyle\sum_{j=1}^{j=l} a_{0j}(-t_j),$$

$$z_i=a_{i0}+\textstyle\sum_{j=1}^{j=l} a_{ij}(-t_j),\ i=1,\ \cdots,\ \rho,$$

where the z_i are the current basic variables and the t_j the current nonbasic variables. (The a_{ij} coefficients of the current tableau are not to be confused with the components of the A matrix.) Upon obtaining an optimal tableau ($a_{i0}\geqq 0$ and $a_{0j}\geqq 0$ for $i, j\geqq 1$), go to Step 2.

2. If a_{i0} is integer for all i, the basic solution $z_i=a_{i0}$ and $t_j=0$ (all i, j) gives a feasible integer solution for (2) by identifying the variables x_j from among those currently designated z_i and t_j. Otherwise, if some a_{i0} is noninteger, adjoin a cut[4,2] and reoptimize with the dual simplex method.

3. If some a_{i0} is still noninteger, let t_s denote the current nonbasic variable that was the slack variable for the cut adjoined (most recently) in step 2. Replace a_{0s} with $a_{0s}-K<0$, where K is an integer (e.g., the least integer $<a_{0s}$). Then reoptimize with the primal simplex method and return to 2.

The purpose of Step 3 is to exploit the fact that the cut slack qualifies to be one of the u_i of the theorem. Thus, it assigns the slack a weight in the objective function designed to drive it basic, thereby possibly modifying, but not discarding, the weights assigned to the other u_i.

Finite convergence is guaranteed if one uses the choice rules of GOMORY[4] in Step 2 and bypasses Step 3 after a fixed number of iterations. The main point, of course, is that the method gives a way to pursue integer feasibility by exploiting the theorem.

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