11

A Study of Alternative Relaxation Approaches for a Manpower Planning Problem†

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This paper examines a variety of relaxation strategies for zero–one integer programming problems, containing from 54 to 2,683 variables, that arise in manpower planning applications. These strategies are compared by a primal criterion, which emphasizes the ability to obtain high quality feasible solutions. This contrasts with the usual dual criterion for comparing relaxations, which emphasizes objective function bounds obtained from solutions that are generally not feasible. The changed emphasis requires a change in the *use* of relaxations, which may be viewed from the standpoint of generating trial solutions for heuristic programming or as a fundamental component of branch and bound. Computer tests show that a combined surrogate-Lagrangean strategy is the most effective for the problems examined followed by a pure surrogate relaxation strategy. All other approaches, including generalized Lagrangean relaxation, fared substantially worse, particularly in terms of solution quality.

1. INTRODUCTION

Manpower planning is an area of growing importance for the effective utilization of human resources. Today, increased recognition is being given to the fact that the assignment of individuals to jobs to make use of their skills while providing adequate job satisfaction is crucial to the smooth and economical function of any large organization. This recognition, coupled with the awareness that mathematical and computer optimization techniques can materially assist in large-scale planning operations, has brought about

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a wealth of research into ways of formulating and solving manpower planning problems [3–6, 9, 17, 26–29].

Advances in mathematical formulations designed to capture significant real-world complexities of the manpower planning problem have been achieved in an evolutionary succession of models, beginning with the CADA system and its offspring [26–29], and culminating most recently in the extended goal programming (EGP) manpower planning model of [17]. This model, which consists of a network quasi-assignment component augmented by two linear side constraints, can be developed as follows.

The classical assignment problem may be stated as

$$\text{minimize} \quad \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{\{j\mid(i,j)\in A\}} x_{ij} = 1, \qquad i\in M, \tag{2}$$

$$\sum_{\{i\mid(i,j)\in A\}} x_{ij} = 1, \qquad j\in N, \tag{3}$$

$$x_{ij} \geq 0, \qquad (i,j)\in A, \tag{4}$$

where $M = \{1, 2, \ldots, m\}$ is the set of men, $N = \{1, 2, \ldots, n\}$ is the set of jobs, A is the set of admissible assignments (arcs), $x_{ij} = 1$ (0) if man i is (is not) assigned to job j, and c_{ij} is the cost of assigning man i to job j.

Many real-world situations have two features which this simple formulation cannot handle satisfactorily. First, the set of admissible arcs A may be such that there are no feasible solutions to constraints (2)–(4). Second, the problem may have multiple competing objectives. The first difficulty can be overcome by adding a dummy man $m + 1$ which can fill any job and a dummy job $n + 1$ which can be filled by any man. This augmented model, called the quasi-assignment model, can be stated as follows:

$$\text{minimize} \quad \sum_{(i,j)\in A'} c_{ij}x_{ij} \tag{1'}$$

$$\text{subject to} \quad \sum_{\{j\mid(i,j)\in A'\}} x_{ij} = 1, \qquad i\in M, \tag{2'}$$

$$\sum_{j\in N'} x_{m+1,\,j} = n, \tag{2a'}$$

$$\sum_{\{i\mid(i,j)\in A'\}} x_{ij} = 1, \qquad j\in N, \tag{3'}$$

$$\sum_{i\in M'} x_{i,\,n+1} = m, \tag{3a'}$$

$$x_{ij} \geq 0, \qquad (i,j)\in A', \tag{4'}$$

where

$$M' = M \cup \{m + 1\}, N' = N \cup \{n + 1\},$$

and

$$A' = A \cup \{(m + 1, j) | j \in N\} \cup \{(i, n + 1) | i \in M\} \cup \{(m + 1, n + 1)\}.$$

Note that (2′)–(4′) always have a feasible solution for any A and that the c_{ij}, $(i, j) \in A' - A$, can be defined to reflect the cost of an unassigned man or unfilled job or to provide for the maximum assignment of men to jobs.

The EGP model is an extension of (1′)–(4′) which accommodates multiple objectives. Specifically, suppose there are three objectives i, $i = 1, 2, 3$, and let $\bar{c}^i$, $i = 1, 2, 3$, be the optimal objective function value obtained when (1′)–(4′) is solved using $c = c^i$. Then the EGP model can be stated as

$$\text{minimize} \quad \sum_{(i, j) \in A} c_{ij}^1 x_{ij} \tag{1′}$$

subject to the quasi-assignment constraints (2′)–(4′)

and

$$\sum_{(i, j) \in A'} \frac{c_{ij}^k}{\bar{c}^k} x_{ij} - \bar{c}^k \leq \alpha_k \left(\sum_{(i, j) \in A'} \frac{c_{ij}^1}{\bar{c}^1} x_{ij} - \bar{c}^1 \right), \qquad k = 2, 3, \tag{5′}$$

where the α_k are weights defining the relative importance of the objectives.

The model is not only relevant to manpower planning but is representative of the class of integer programming problems with major imbedded network structures that have appeared in wide varieties of real-world applications (see, e.g., [3, 6, 17, 18, 26, 30]).

1.1. Purpose and Scope of Investigation

The goal of this paper is to determine strategies for solving these manpower planning problems effectively. Advances toward this goal almost inevitably have implications for solving other members of this problem class and thus provide the foundation for algorithmic investigations beyond the domain of our present concern.

The multiobjective manpower planning problem, whose origins and nature are more fully described in [6, 17], may contain almost a half million variables and 2400 constraints (disregarding bounds on variables). The computational testing reported in this paper was performed on the set of

15 EGP problems reported in [17]. These problems range in size from 20 constraints and 54 variables up to 135 constraints and 2,683 variables. Integer programming problems of this size would normally be viewed as extremely difficult, perhaps even impossible, to solve (within a practical time limit). However, because of the large imbedded network structure, a more optimistic outlook is warranted.

The primary concern of this paper is to identify ways to exploit the network-related form of this problem. More particularly, our goal is to identify an effective solution strategy to be incorporated within branch and bound (B & B) and related or extended methods such as parametric B & B [13]. Numerous investigations into combinatorial problem solving have emphasized the value of relaxation strategies in B & B, and thus, we focus on relaxation strategies acclaimed to be successful in the literature, as well as several supplemental strategies that have been made the basis of integer programming solution routines. However, we also examine a relaxation strategy (in two variants) that has so far escaped practical study. One of our principal discoveries is that this strategy, which is a composite of surrogate constraint and generalized Lagrangean relaxations, proves to be far more effective than any of the other approaches examined.

1.2. A Primal Orientation

It should be stressed that we favor the use of a different (or, more precisely, an additional) measure of the effectiveness of a relaxation strategy than emphasized in previous studies. Whereas customary measures, both theoretical and pragmatic, have concentrated on a *dual measure* of effectiveness (by reference to optimistic bounds on the objective function), we choose to call attention to the importance of a *primal measure* of effectiveness (by reference to the capability to generate feasible trial solutions).

This altered emphasis has not been undertaken before, primarily because the notion of a problem relaxation is the cornerstone of duality theory. Indeed, customary uses of relaxations have been characterized as "duality exploiting" [16]. Yet it is repeatedly conceded that one of the prime virtues of B & B is the ability to obtain good feasible solutions and users of B & B solution routines look for this ability, especially in their attempts to solve large problems, when integer programming algorithms are often unable to verify optimality. Since the verification of optimality is a dual-related function (employing objective function bounds) and the identification of good feasible solutions is a primal-related function (employing "solution completions"), the inclusion of a primal orientation in evaluating problem relaxations would seem to be particularly justified. This is notably so for parametric B & B, which has an even stronger connection to primal solution strategies.

This change of emphasis to a primal orientation, however, harbors a few difficulties. Except under special circumstances, it is generally necessary to go beyond the solution to a relaxed problem to obtain a trial solution that has a chance of being feasible for the original problem. Thus, for example, the solution to an LP problem must, at the very least, be rounded in order to obtain an integer-valued trial solution to the integer problem. Consequently, the introduction of a primal solution criterion for a problem relaxation, at the very least, carries with it the burden of specifying how to replace a fractional solution by an integer solution when the integer restriction is not automatically satisfied. More complex types of "solution completion" may also require strategies to come closer to satisfying other types of problem constraints (see, e.g., [15]).

This changed emphasis has further ramifications; it also necessitates a change in the characterization of a good relaxation. Parameter values that yield a good relaxation in a dual sense may be substantially different from those that yield a good relaxation in a primal sense. The findings of our study show that this happens in the majority of cases. Such a possibility has escaped previous investigations that have focussed wholly on the dual evaluations of relaxation strategies. The implications of these findings are clearly profound: If we seek improved feasible solutions, then the relaxation strategies based on duality considerations are inappropriate. This is not to say duality considerations are without value. Rather, they should be supplemented by other considerations to obtain a more effective overall strategy.

The present study demonstrates that combined surrogate-Lagrangean strategies dominated all other solution strategies we investigated, according to a primal solution criterion. Further, these strategies succeeded as well as they did, not because the problems studied were easy to solve (i.e., not because good feasible solutions were easy to find by the customary approaches), but in spite of the failure of traditional alternatives. The power of the primal-related relaxation strategies is illustrated by the fact that we were able to obtain good feasible 0–1 solutions at the root of the B & B tree (before branching to deeper levels) in a total elapsed CPU time of 150 to 380 seconds for the *entire* collection of 15 problems. Since it is rare to obtain good feasible integer solutions from a root node, and since the other methods fared much less successfully, the remarkable speed at which the feasible solutions were obtained indicates a major advance in solution strategies for this class of problems.

To provide a foundation for understanding the uses we have made of problem relaxations, and the specific results obtained by our testing, we sketch the essential concepts of problem relaxations in Section 2.

2. PROBLEM RELAXATION

The problem

$$\begin{aligned} &\underset{x \in X}{\text{minimize}} \quad g(x) \\ &\text{subject to} \quad G(x) \leq 0 \end{aligned} \tag{R}$$

is said to be a *relaxation* of the problem

$$\begin{aligned} &\underset{y \in Y}{\text{minimize}} \quad f(y) \\ &\text{subject to} \quad F(y) \leq 0 \end{aligned} \tag{P}$$

if the following conditions hold.

(1) If x^* is optimal for (R) and y^* is optimal for (P), then $g(x^*) \leq f(y^*)$ (bounding condition).

(2) There is a transformation $T(x) = y$ such that if x^* is optimal for (R) and $y^* = T(x^*)$ is feasible for (P) [satisfies $y^* \in Y$ and $F(y^*) \leq 0$], then $f(y^*) = g(x^*)$ implies y^* is optimal for (P).

(3) If (R) has no feasible solution, then (P) has no feasible solution. [This condition may be viewed as an instance of condition (1) under the convention that $g(x)$ and $f(y)$ are minimized at infinity when the constraint sets of (R) and (P) are empty.]

The motive for calling (R) a relaxation of (P) is that the foregoing conditions are satisfied most simply when $x = y$, the constraint region of (R) is less restrictive (more "relaxed") than that of (P), and $g = f$ [or else g is a simple underestimating function that yields $g(x) \leq f(x)$ for all x that are feasible for (P)]. Many, but not all, of the relaxations that have been studied extensively or used in practical applications (including those investigated here) are of this simple form. Nevertheless, their theoretical content and power is extensive. Indeed, the relaxations we employ in this study give rise to remarkably rich duality results and have intriguing consequences when incorporated into primal strategies.

The intimate connection between problem relaxations and nonlinear duality theory can be expressed as follows. As noted in [16], all duality theories arise by defining the dual of (P) to be the problem of finding the *strongest* (R) (from the class of relaxations under considerations), where (R) is defined parametrically. Thus, for example, if the functions composing (R) are expressed as

$$\begin{aligned} g(x) &= ug_0(x) \qquad u \in U, \\ G(x) &= wG_0(x) \qquad w \in W, \end{aligned}$$

where u and w are vectors or matrices of parameters (some constant, some variable, depending on the limiting conditions of $u \in U$ and $w \in W$), then the problem of finding the strongest relaxation (R) is that of determining $u^* \in U$ and $w^* \in W$ to yield the tightest bound for (P) by condition (1). Namely, the dual of (P) is defined as

$$\underset{u \in U,\ w \in W}{\text{maximize}} \begin{pmatrix} \min_{x \in X} ug_0(x) \\ wG_0(x) \leq 0 \end{pmatrix}$$

(replacing maximum and minimum by supremum and infimum for the general case). Viewed in the form of a "maximin" problem, the dual seems rather tricky; viewed as a problem of finding the relaxation that yields the tightest objective function bound for (P), the dual seems more intuitively reasonable.

From a pragmatic standpoint, the dual problem is that of finding the relaxation (from the class considered) whose solution is most likely to solve the original problem (P), at least by providing a strong bound. Every relaxation gives an *optimistic estimate* of the minimum $f(y)$ for problem (P). A weak relaxation gives a grossly optimistic estimate, a strong relaxation gives a more accurate estimate, and the study of strongest relaxations is what duality theory is all about. [That is, duality theory seeks to characterize the conditions that indicate optimality for the dual and to express the relation between optimal solutions to the primal (P) and the strongest relaxation (R).]

Beyond the realm of theoretical significance, as embodied in duality theories, problem relaxations have turned out to have major practical consequences for solving nonlinear and combinatorial optimization problems, particularly those of integer programming. In practical applications, it is natural to seek a relaxation that is *good*, which is most often interpreted to mean

(i) it should be relatively strong, in the bounding sense,
(ii) it should be easy to solve by comparison to (P).

These criteria for a good relaxation are, of course, not clearly defined, except in terms of a simple dominance scale, whereby one relaxation can be called better than another if it yields a stronger bound and is also easier to solve. The weakness in this type of characterization of goodness has not occasioned much difficulty, however, because the ultimate pragmatic test has always been to imbed a relaxation strategy (i.e., a subroutine for generating and solving problem relaxations) within a global solution method—such as branch and bound—and to determine whether the method works better with or without the relaxation strategy (as measured by such things as speed of obtaining optimal and near-optimal solutions, and reliability over the class of problems examined). Experience has shown that relaxation strategies indeed markedly improve solution efforts for many types of optimization problems (see, e.g., [7, 8, 10, 21, 24]).

2.1. Two Major Relaxation Approaches

Two types of problem relaxations, each of which has given rise to a duality theory and has proven especially valuable in practical settings, are the *generalized Lagrangean* and the *surrogate constraint* relaxations. The generalized Lagrangean relaxations, most notably, have found use in a wide spectrum of combinatorial applications of scheduling and planning.

The generalized Lagrangean relaxation has the form

$$\underset{y \in Y_0}{\text{minimize}} \quad f(y) + \lambda F(y), \tag{LR}$$

where Y_0 is a superset of Y and λ is a vector of nonnegative parameters. (The dual problem is thus to find $\lambda \geq 0$ to maximize the minimizing value over y, thereby yielding the tightest bound and strongest relaxation.)

The surrogate constraint relaxation on the other hand has the form

$$\begin{aligned} &\underset{y \in Y_0}{\text{minimize}} \quad f(y) \\ &\text{subject to} \quad wF(y) \leq 0, \end{aligned} \tag{SR}$$

where w is a vector of nonnegative parameters (as is λ). In the case where more than one constraint is allowed to replace the system $F(y) \leq 0$, w is a matrix of non-negative parameters.

Almost completely unexamined in prior studies is the combined surrogate-Lagrangean relaxation

$$\begin{aligned} &\underset{y \in Y_0}{\text{minimize}} \quad f(y) + \lambda F(y) \\ &\text{subject to} \quad wF(y) \leq 0, \end{aligned} \tag{SLR}$$

although the theory of such relaxations, in the setting of nonlinear duality, is well developed [14, 19].

In each of these three types of relaxations, the constraining relation $y \in Y_0$ is generally taken to be one over which it is relatively easy to obtain a minimizing solution. Hence in the manpower planning context $y \in Y_0$ is quite naturally taken to be the constraint condition defining the quasi-assignment network portion of the problem. The inequality $F(y) \leq 0$ then conveniently summarizes the side constraints of the manpower planning problem. Note that the stipulation that Y is the network constraint region restricted to 0–1 solutions causes Y_0 to be a superset of Y.

Typically, for most types of problems, the Lagrangean relaxation (LR) is easier to solve than the surrogate relaxation (SR). Also, theory suggests that the values of λ that give strong Lagrangeans are likely to be easier to find than the values of w that give strong surrogates, though little prior experience exists on this matter. Balancing this is the fact that surrogate relaxations are stronger than Lagrangean relaxations and have smaller duality gaps. That is, a solution to a surrogate problem is more likely to solve the original problem than a solution to a Lagrangean, when the strongest version of each is used.

Thus, as a general rule, if a Lagrangean relaxation (LR) works well for a particular class of problems, it is probably the method of choice, since (at least for most of today's solution technology) it is easier to implement. On the other hand, if the Lagrangean relaxation does not work well (i.e., does not give rise to a strategy that renders the original problem readily solvable), then the surrogate relaxation (SR), or even the combined surrogate-Lagrangean relaxation (SLR), may prove valuable.

This "rule of thumb" guideline for determining when to use a particular relaxation applies to the more customary uses of such relaxations in a *duality* setting. We shall now discuss uses of these relaxations in a primal setting.

3. RELAXATIONS AND PRIMAL STRATEGIES

In order to use the (LR), (SR), and (SLR) relaxations in a primal strategy it is necessary to convert the solution for $y \in Y_0$ (which satisfies the network quasi-assignment constraints of the manpower planning problem) to a solution in which $y \in Y$ (which also satisfies the 0–1 conditions). There is no difficulty doing this for the Lagrangean (LR), since in the absence of additional constraints, the network solutions to $y \in Y_0$ will automatically be integer (for integer data, using standard methods), and no additional effort is required.

The surrogate and surrogate-Lagrangean relaxations (SR) and (SLR) pose an additional problem, since by including the constraint $wF(y) \leq 0$ they consist of network problems subject to a side constraint, and solutions to this problem characteristically are not integer valued. (For the problems we studied, the number of noninteger valued variables for relaxations with a side constraint generally ranged from 3 to 7.) Based on the study of [18], we elected to modify the noninteger solution for such a problem by executing a single pivot to bring the slack variable for $wF(y) \leq 0$ into the basis. The resulting solution is an extreme point of the network constraint region and therefore integer. With only this modification we obtained trial solutions to be compared with those obtained from the (LR) relaxation, and which we plugged into the original problem (P) to be checked for feasibility.

As noted earlier, while a rigorous definition of the *strength* of a relaxation in a dual setting is easily supplied, a definition of the *goodness* of a relaxation is considerably murkier (though this seems not to cause any practical difficulty). The same is true in the primal setting. More specifically, we would loosely stipulate that a relaxation is good in the primal sense if, in addition to being relatively easy to solve, it can give rise to a trial solution (by a suitable modification of its own solution) that has a high probability of being feasible and near optimal. We particularly value near-optimal solutions obtained for subproblems of a B & B enumeration tree at nodes *close to the root*.

The liberal use of qualitative and imprecise terms in this description allows, as in the dual setting, only a simple type of dominance condition to emerge with any rigor; for example, one relaxation may be termed better than another (from a primal standpoint) if it yields a greater number of near-optimal solutions at earlier nodes of a B & B tree than another relaxation. This vagueness occasions no embarrassment for problem solvers, however. The intuitive thrust is clear, and the ultimate test is whether imbedding such a strategy in a global solution algorithm yields better solutions at earlier stages of calculation.

For this reason, in our tests of different approaches by the primal orientation, we have elected to face the hardest challenge in seeking trial solutions: all solutions were generated only for the problem at the root of the B & B tree (hence for the original "unbranched" problem). Although this reduced the chances of finding feasible solutions (and certainly of finding good feasible solutions), any successes obtained could be viewed as significant. The special structure of the manpower planning problem would seem to enhance the likelihood of generating good feasible trial solutions, yet as our results show, by employing tests of other methods that fared much less effectively than the front runners, this enhanced likelihood seems nonexistent or negligible for standard procedures. The significance of these remarks will become clearer upon describing the full range of test procedures and their outcomes in detail.

4. ALTERNATIVE STRATEGIES TESTED

To provide a solid basis for comparison, and to determine the general level of problem difficulty, we have undertaken to test a variety of trial solution approaches, both from classes that are viewed as relaxation strategies (as previously characterized) and from other classes as well. The various approaches are itemized below.

Table 1

Problem Specifications

Problem	Men	Jobs	Arcs	$\bar{c}$	$\bar{d}$	$\bar{u}$
	15	12	105	1974	5342	2711
2	21	12	175	216	7312	771
3	9	9	41	1049	908	3014
4	27	9	117	142	1254	2028
5	30	18	280	121	8390	4329
6	9	9	51	2806	3246	3584
7	21	14	205	167	457	4499
8	74	41	1207	11231	23009	14387
9	90	22	697	6938	13298	6893
10	86	47	2549	264	19622	8555
11	19	7	66	1215	3275	1556
12	24	17	51	14221	14904	14767
13	44	33	363	22917	28704	21997
14	18	14	105	5470	8619	7254
15	13	5	35	1948	1357	958

A set of 15 EGP problems described in Table 1 were employed in the testing which was performed. These problems represent actual Navy personnel rotation assignment decisions [17] and have three objectives: c, the dollar cost of making an assignment; d, the desirability to the man of an assignment; and u, the utility to the Navy of an assignment. Table 1 also gives the optimal values of these objectives considered independently (i.e., $\bar{c}$, $\bar{d}$, and $\bar{u}$). In the following, the dollar cost objective c was always considered to be the primary one appearing in the objective function (1′) and in the right-hand side of the two extra nonnetwork constraints (5′). In order to facilitate comparison of the relative effectiveness of the strategies tested, we present in Table 2 the best feasible integer solutions available (from any method) for these problems. For each problem the values of the objectives cx, dx, and ux are given. Subsequent tables give the same information about integer solutions obtained from the various strategies to be described and, for the feasible integer solutions, give the percent deviation from the best solution in Table 2 in terms of the primary objective cx.

4.1. Vertex Ranking Method

This method can be applied as a complete algorithm (see, e.g., [30]) or as a means of generating trial solutions. Basically it involves two steps starting at an optimal extreme point of the continuous (LP) relaxation of the 0–1 problem: (a) Examine all adjacent extreme points, ranking them in

Table 2

Best Feasible Integer Solutions

Problem	*cx*	*dx*	*ux*	Problem	*cx*	*dx*	
1	2870	5342	2984	8	12255	23009	15414
2	256	8259	863	9	7024	13338	6919
3	1923	1368	3139	10	313	22985	10125
4	175	1254	2098	11	1658	3363	1730
5	138	9365	4890	12	14918	15106	15228
6	3085	3246	3763	13	23410	28704	22253
7	1100	457	5544	14	5607	8726	7419
				15	380	2044	1251

increasing order of their objective function value; (b) Select the lowest ranked point not previously chosen. If the latter is integer feasible it is optimal; otherwise return to step (a) to incorporate the adjacent extreme points of this newly selected point in the ranking. Various refinements are possible to streamline the approach and help avoid multiple examinations of a given point, though these are not critical to the method as a strategy for generating integer trial solutions.

Quite briefly, the vertex ranking approach failed for the problems studied because of the very large number of adjacent extreme points and the fact that most of these, due to degeneracy, had the same objective function value as the optimal continuous solution, *yet were all noninteger*. It was far too time consuming to undertake examination of the full range of adjacent vertices looking for an integer solution, thus rendering the approach impractical.

4.2. Linear Search

The linear search approach is a heuristic procedure due to Hillier [22] and tested in the B & B setting by Jeroslow and Smith [23]. Its implementation is based on rounding points that lie on a line segment determined by reference to a simplex whose edges coincide with those through the adjacent extreme points. The very large numbers of these edges, and the necessity to perform recovery calculations on each from data that is not explicitly available in the LP tableau (due to the use of network basis compactification for economy of memory and efficiency of solution) turns out to be wholly impractical for the problems under consideration. Consequently, in contrast to the other procedures cited, this method was merely studied without undertaking a computer implementation, and abandoned on the basis of the resemblance of its calculations to the determination of adjacent extreme points required by the vertex ranking approach. A possible variant of this approach would be to construct a search line by reference to only a subset of the edges, though the value of this is unclear since it places the search line on a substantially lower dimensional face of the simplex. Furthermore, the efficacy of rounding is somewhat questionable due to the multiple-choice structure of the quasi-assignment problem.

4.3. Restricted Basis Entry

The restricted basis entry approach has been used most frequently as an approximating solution method for piecewise linear problems in nonlinear programming, and has also been applied to 0–1 integer programming problems by Berman [2]. Due to the large network component of the problem, it seems plausible (in spite of the difficulty of finding integer vertices in the vertex ranking scheme) that once a feasible integer vertex is identified, it may be possible to progress to improved feasible integer solutions by allowing only basis exchanges that are integer feasible. Our tests disclosed otherwise. In fact, our tests of this method beginning from integer solutions obtained from the surrogate-Lagrangean approach (discussed later in Section 4.11) failed in every case to find an integer solution better than the starting solution.

4.4. Solution Testing of Quasi-Assignment Extreme Points

The solution of the quasi-assignment network problem, independent of the side constraints, involves the examination of several hundred integer points. A colleague proposed to us the likelihood that a number of these points would be feasible for problem (P), and that some of these points would yield rather decent objective function values. Accordingly, we tested this hypothesis for each of the manpower planning problems. The outcome was entirely negative. Feasible solutions were found for six of the problems, and those found were without exception very poor. At this stage of the testing we were convinced that the problems we dealt with were anything but trivial, and out next sets of tests reinforced this conclusion.

4.5. Surface Optimization

This approach is an instance of multiobjective and goal programming which is strongly suggestive of generalized Lagrangean relaxation. Surface optimization takes the slacks for the side constraints into the objective function (in conjunction with a slack for the original objective function) in

Table 3[a]

Surface Optimization

Problem	cx	dx	ux	% from best
	Priority ordering $c - d - u$			
1	1974	7004*	3688*	inf
2	216	9367*	1426*	inf
3	1049	1807*	3131*	inf
4	142	4327*	2358*	inf
5	121	10004*	5192*	inf
6	2806	3246	4608*	inf
7	167	4952*	5383*	inf
8	11231	25581*	16921*	inf
9	6938	14566*	7872*	inf
10	264	24222*	12913*	inf
11	1215	3807*	2352*	inf
12	14221	16005*	15912*	inf
13	22917	28704	22791*	inf
14	5470	8726*	7429*	inf
15	1948	2255*	3450*	inf
	Priority ordering $d - u - c$			
1	2870	5342	2984	0
2	2006	7312	1190	683.59
3	1070	908	3466*	inf
4	175	1254	2098	0
5	1045	8390	5083	657.25
6	2806	3246	3763*	inf
7	1100	457	5544	0
8	11315	23009	15391*	inf
9	7891	13298	7144*	inf
10	323	19622	11437*	inf
11	2097	3275	1556	26.48
12	14977	14904	14894	0.19
13	22917	28704	21997*	inf
14	6099	8619	7783	8.77
15	2035	1357	1691*	inf

[a] inf—no feasible integer solution was found. *—extra constraint corresponding to this objective was violated.

a strictly hierarchical or preemptive fashion. This approach was tested for generating trial solutions primarily because it has been regarded in the past as one of the more useful approaches for dealing with manpower planning problems.

The results of applying surface optimization according to different priority sequences are indicated in Table 3. In sum these results show that the past enchantment with surface optimization was out of place, at least from a primal standpoint. The worst variant, which ranked the objectives in the order *c*, *d*, *u*, failed to obtain any feasible solutions. The best variant, which ranked the objectives in the order *d*, *u*, *c*, obtained feasible solutions for only eight of the problems, and these solutions, in terms of the objective *c*, are rather poor by comparison to those obtained by the better methods.

4.6. Generalized Lagrangean Using Optimal Dual Weights

This is the classical, most widespread use of Lagrangean relaxation in B & B, by which trial solutions from optimal dual weights (i.e., those yielding the strongest relaxation) are plugged into the original problem (P). (In

Table 4[a]

Generalized Lagrangean with Optimal Weights

Problem	*cx*	*dx*	*ux*	% from best
	2074	6241*	3354	inf
	276	9157	787	7.81
	1081	908	3466*	inf
4	175	1254	2253	0
5	138	9365	4890	0
6	2896	3246	3763*	inf
7	1209	457	6090	9.91
8	12339	23547	14664	0.65
9	7002	13448*	6917	inf
10	313	22985	10125	0
11	1694	3363	1600	2.17
12	14918	15106	15228	0
13	23496	28704	22083	0.37
14	5638	8726	7419	0.55
15	2846	2223*	958	inf

[a] inf—no feasible integer solution was found. *—extra constraint corresponding to this objective was violated.

subgradient search approaches, trial solutions to relaxations identified en route to solving the dual are also used as trial solutions, and these are considered subsequently.) Table 4 shows that the strongest Lagrangean relaxation (in the dual sense) yielded feasible trial solutions to only 10 of the problems, and that the quality of these feasible solutions was generally not as good as those obtained by other methods.

4.7. Generalized Lagrangean Using *a Priori* Intuitive Weights

Although standard applications of Lagrangean relaxations generate trial solutions from the strongest relaxation, as in Section 4.6, the possibility that primal solution capability may result from alternate parameter choices prompts the investigation of the weighting scheme that was determined on the basis of intuitive argument [26] to be best for the manpower planning problem. Table 5 shows this approach obtained feasible solutions for eight of the problems, and that the quality of these solutions is generally inferior, and in some cases much worse, than those obtained by the better strategies.

Table 5[a]

A Priori Weighting

Problem		dx	ux	% from best
	2947	5342	3096	2.68
2	303	7851	1142*	inf
3	1072	908	3468*	inf
4	196	1254	2200	12.00
5	193	8734	5095	39.86
6	2896	3246	3763*	inf
7	228	1356*	5184	inf
8	11386	23547*	14706*	inf
9	7028	13338	6896	0.06
10	372	19622	11757	18.85
11	1694	3363	1600	2.17
12	15034	14904	15228	0.78
13	23496	22083	28704	0.37
14	5470	8726*	7429*	inf
15	2035	1357	2424*	inf

[a] inf—no feasible integer solution was found. *—extra constraint corresponding to this objective was violated.

4.8. Generalized Lagrangean Using Weights from Subgradient Search

As already noted, standard dual procedures based on subgradient search generate trial solutions en route to determining optimal dual weights. The full collection of trial solutions generated in this fashion was examined for feasibility and the best recorded. The results tabulated in Table 6 show that feasible solutions were obtained for 13 of the problems. In quality, these solutions are generally the same as those obtained using the weights of Section 4.6 although one of the new feasible solutions (to problem 15) is extremely poor. Only four of the solutions tied in quality with those obtained by the better approaches.

Table 6[a]

Generalized Lagrangean Using Interval Weights

Problem	cx	dx	ux	% from best
	2947	5342	3096	2.68
	276	9157	787	7.81
	1934	908	3537	0.57
	175	1254	2253	0
	138	9365	4890	0
	2896	3246	3763*	inf
	1209	457	6090	9.91
8	12339	23547	14664	0.65
9	7002	13448*	6917	inf
10	313	22985	10125	0
11	1694	3363	1600	2.17
12	14918	15106	15228	0
13	23496	28704	22083	0.37
14	5638	8726	7419	0.55
15	4323	2223	1251	1037.63

[a] inf—no feasible integer solution was found.
*—extra constraint corresponding to this objective was violated.

4.9. Surrogate Relaxation Using Optimal Dual Weights

The surrogate relaxation defines a dual problem, as does the Lagrangean, in the manner indicated in Section 2. Surrogate duality theory [14, 19] however, discloses that the optimal weights for the surrogate may be different from those of the Lagrangean, and are characterized by different relationships. In particular, the dual functional for the surrogate problem is quasi-concave, and due to the shorter history of application of surrogate duality, no extensive body of empirical research yet exists for identifying the most efficient ways to search for optimal dual surrogate weights. Consequently, we employed a subgradient procedure resembling the standard approach for the Lagrangean. This procedure worked, but clearly was not ideally tailored to the surrogate dual functional. The most notable difficulty of this type of adaptation was the determination of step sizes. This discovery, we believe, pinpoints a key direction for future research in this area.

Having found optimal dual surrogate weights by our adapted subgradient procedure, we utilized the *slack pivot* technique described in Section 3 for obtaining an integer trial solution. One further point should be noted. The surrogate dual, and hence the subgradient search, was defined by letting Y_0 consist of the network semiassignment region, excluding the 0–1 integer condition (as specified in Section 3). This composition of Y_0 makes no difference in the case of the generalized Lagrangean, since optimization over Y_0 is the same as optimization over Y (which includes the 0–1 condition), as a result of the integer extreme point property of the network. However, contracting Y_0 to Y makes a good deal of difference in the surrogate relaxa-

Table 7[a]

Surrogate Relaxation Using Optimal Weights

Problem	cx	dx	ux	% from best
1	2892	6105	2823	0.77
2	254	7861	1223*	inf
3	1923	1368	3139	0
4	175	1254	2253	0
5	148	10264*	4645	inf
6	2922	3803*	3585	inf
7	1100	457	5879	0
8	12301	23162	15105	0.34
9	7021	13263	7167*	inf
10	315	23478*	9963	inf
11	1694	3363	1600	2.17
12	14918	15106	15228	0
13	23293	29603*	22083	inf
14	5778	9625*	7254	inf
15	2846	1357	1691*	inf

[a] inf—no feasible integer solution was found. *—extra constraint corresponding to this objective was violated.

tion, as evidenced by the fact that the surrogate solutions were generally noninteger (usually containing from 3 to 7 fractional-valued variables). In view of the importance of this difference reflected in our findings, we anticipate that it would be more valuable to employ a surrogate search strategy by reference to integer trial solutions rather than continuous solutions, using a heuristic such as the slack pivot technique at each level of the search. (This accords with the strategy for surrogate constraint determination originally proposed in [11].)

Results from using the surrogate relaxation defined relative to continuous solutions are shown in Table 7. In this instance, feasible solutions were obtained for seven of the problems, and the solution quality for these feasible solutions was generally quite good, particularly in view of the fact that only a single weight vector was tested for each problem.

4.10. Surrogate Relaxation Using Interval Weights

Results obtained from using the slack pivot heuristic with the surrogate relaxation noted above indicated that surrogate constraints whose weights differed from the optimal dual weights might produce good integer trial

Table 8[a]

Surrogate Relaxation Using Interval Weights

Problem	cx	dx	ux	% from best
1	2870	5342	3688	0
2	256	8259	863	0
3	1925	1368	3139	0.10
4	175	1254	2253	0
5	138	9365	4890	0
6	3085	3246	3763	0
7	1100	457	6231	0
8	12281	23997	14485	0.18
9	7024	13338	6919	0
10	313	22985	10125	0
11	1658	3363	1730	0
12	14918	15106	15228	0
13	23496	28704	22083	0.37
14	5607	8726	7419	0
15	2846	2223*	958	inf.

[a] inf—no feasible integer solution was found.
*—extra constraint corresponding to this objective was violated.

4.11. Surrogate-Lagrangean Relaxation Using Simple Partitioning and Interval Weights

The combined surrogate-Lagrangean relaxation, in which the side constraints are both transformed into a surrogate constraint and absorbed into the objective function, was tested in two ways. Both of these ways used a highly simplified determination of the weights for the constraints as they were parameterized in the surrogate constraint and objective function. In view of this, it may seem surprising that this strategy worked so effectively. However, in retrospect we believe this is due to the fact that the two side constraints were equally important in constraining the feasible region in the vicinity of integer optima.

The first and simplest application of the surrogate-Lagrangean strategy merely assigned one of the two constraints to be the surrogate constraint (giving its companion a weight of 0) and took the remaining constraint into the objective function. This accords with the partitioning strategy proposed in [14], whereby the constraints are divided into groups in creating a surrogate-Lagrangean relaxation.

To further simplify the approach, drawing on earlier findings that the best integer trial solutions could arise from parameterizations that bore no apparent relationship to the optimal dual parameterization, we varied the weight on the constraint in the objective function over an interval, ignoring the dual functional, and generating a trial solution by the slack pivot technique at each point. The results are summarized in Table 9. Note that this approach proved demonstrably superior to all those previously tested, including both the Lagrangean and the surrogate approaches used in isolation. Feasible integer solutions were found for all of the problems, and the quality of these solutions equals or dominates that of the other procedures for all except four of the problems solved in common. In these latter cases, the solutions were only slightly poorer than those obtained by the surrogate approach with interval weights. One of the findings of the combined surrogate-Lagrangean approach, which disclosed the "equally important"

Table 9

Surrogate-Lagrangean Using Simple Partitioning

Problem	cx	dx	ux	% from best
1	2870	5342	3539	0
2	256	8259	863	0
3	1925	1368	3139	0.10
4	175	1254	2253	0
5	138	9365	4890	0
6	3085	3246	3763	0
7	1100	457	6231	0
8	12259	23009	15414	0
9	7024	13338	6945	0
10	320	22492	10203	2.24
11	1658	3363	1730	0
12	14918	15106	15228	0
13	23496	28704	22083	0.37
14	5623	8726	7419	0.29
15	380	2044	1251	0

influence of the constraints alluded to earlier, was that no significant difference resulted when the identities of the constraints incorporated into the objective function and the surrogate constraint were switched.

4.12. Surrogate-Lagrangean Relaxation Using Strong Surrogates and Interval Weights

The second form of surrogate-Lagrangean relaxation that we tested made use of a more complex determination of the surrogate constraint but retained the simplified Lagrangean strategy of taking only one of the two constraints into the objective function. Retrospective analysis suggests that this type of strategy is subject to a shortcoming. By first giving one of the constraints a weight in the surrogate constraint, and then taking the same constraint into the objective function (without allowing either subsequent adjustment of the parameters of the surrogate constraint or a "balancing" inclusion of the alternative constraint in the objective function), the single constraint incorporated into the Lagrangean has a lopsided influence. (This was avoided in the simple partitioning scheme which pitted the two constraints directly against each other in their Lagrangean and surrogate roles.) Nevertheless, this strategy worked better than all of the other strategies except for the simple surrogate-Lagrangean partitioning strategy and the

Table 10[a]

Surrogate-Lagrangean Using Strong Surrogates

Problem	cx	dx	ux	% from best
1	2870	5342	3539	0
2	256	8259	863	0
3	1923	1368	3139	0
4	175	1254	2253	0
5	138	9365	4890	0
6	3085	3246	3763	0
7	1100	457	5879	0
8	12290	23997	14486	0.25
9	7891	13298	7872*	inf
10	336	21506	10666	7.35
11	1658	3363	1730	0
12	14918	15106	15228	0
13	23410	28704	22253	0
14	5607	8726	7419	0
15	2846	2223*	958	inf

[a] inf—no feasible integer solution was found.
*—extra constraint corresponding to this objective was violated.

surrogate relaxation using interval weights. The results given in Table 10 show that this method obtained feasible solutions for thirteen of the problems and that eleven of these matched the best found by any method.

5. CONCLUSIONS

Our experience with fifteen test problems from a real-world setting, containing up to 2,683 zero-one variables, indicates that good feasible solutions can be obtained by surrogate and especially by combined surrogate-Lagrangean relaxation. A simple partitioning strategy for the combined surrogate-Lagrangean relaxation succeeded in obtaining feasible solutions for all test problems, which was not accomplished by any other approach (though the surrogate relaxation succeeded in obtaining feasible solutions to all but one problem). By constrast, other strategies not only failed to obtain as many feasible solutions, but failed to obtain solutions of comparable quality. (For example, generalized Lagrangean relaxation obtained only four solutions whose quality matched that of solutions obtained by the better approaches, yet generalized Lagrangean relaxation proved superior to all but the two front running strategies.) Other "nonrelaxation" strategies, such as vertex ranking and restricted basis entry, proved utterly useless for the problems examined, further emphasizing the merits of the successful strategies.

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