Netform Modeling and Applications for Quantum Bridge Analytics

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Abstract

Network-related formulations, known as *netforms*, are becoming increasingly important for expanding the domain of practical applications addressed by Quantum Bridge Analytics. This note briefly describes the nature of a variety of netform models, using illustrative diagrams that disclose the visual aspect that gives these models wide applicability. The diversity and flexibility of these models makes them ideally suited for joining the realm of important model types embraced by Quantum Bridge Analytics.

Background and Origins

Due to perfect storm of converging influences, the Quadratic Unconstrained Binary Optimization (QUBO) model has become the focus of many groups in quantum computing. The privileged position of the QUBO model is coming to be challenged, however, as forays are beginning to appear within the quantum computing community, still fledgling at present, with the goal of tackling other optimization models.

Netform (network-related formulation) models have a broad applicability that makes them an ideal candidate to gain prominence among the next generation of optimization models at the focus of quantum computing. The range of important practical applications embraced by netforms is at least as great as encompassed by QUBO formulations, and the problem-solving technology underlying netform models accords well with the Quantum Bridge Analytics (QBA) perspective, which builds on the theme of the National Academies of Science, Engineering and Medicine to observe that more powerful algorithms within the classical domain are important to set the stage for classical-quantum hybrids. As argued in the National Academies report "Quantum Computing: Progress and Prospects" (2019), such hybrids are not only valuable in the present but will play an important role as quantum computing becomes more mature.[1] (All references cited in this document are accompanied by links in the References section for accessing their sources.)

This overview of netforms is intended to disclose the potential for creating more advanced models and algorithms, building on a rich background of innovations that have established the value of netform technology across a broad spectrum of applications. Practical experience demonstrates that netform modeling and solution strategies have the potential to overcome many of the difficulties in conceptual design and problem solving for system optimization. Moreover, netform technology provides decision-planning tools that can be adapted to a wide range of optimization settings and are easy to understand and apply by practitioners.

Interestingly, the term "netforms" has migrated beyond its origins and has been promoted in the virtual reality domain with a meaning unrelated to optimization. For the reader interested in delving further, netform models in the optimization context are covered in the book Glover et al. (1992) which contains detailed references for the many of the applications illustrated below. Additional background information can be found in Lev (1993) and modern application of netform modeling linked to QBA and with applications to asset exchanges in blockchains appears in Glover et al. (2020).

It is noteworthy that the foundations for the applications of netform models and algorithms were established three decades ago. Nevertheless, their full impact has only recently begun to be felt, due to innovations and advances in computer implementation that have followed. (A similar trajectory has been followed by applications of modern linear and integer programming, whose foundations were established still earlier in the 1950s and 1960s.) This impact stands to be greatly magnified by the association of netform technology with quantum-based computing through Quantum Bridge Analytics.

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[1] A fuller description of the implications of the National Academies Report for QBA appears in Glover et al. (2019).

The following material provides a background for understanding the value of netform modeling and solution technology in the QBA setting by describing a variety of important applications where netforms arise.

The Utility of Netform Models and their Applications

Computer implementation and problem representation have profited from network-related optimization chiefly because advances in this field have established an intimate relationship between problem solving and the identification and exploitation of structure. The development of models based on characterizing structure for the purpose of insight and more effective solution has (1) expanded awareness among optimization theoreticians and practitioners of the range of problems that can be formulated as networks and generalized networks; (2) identified ways to express problems—and key components of problems—in a network-related format, especially in contexts that do not immediately suggest this possibility; (3) introduced flexible notations to identify additional constraining conditions, such as "all-or-none" and "multiple choice" flow restrictions (and integer restrictions in generalized networks); (4) developed a repertoire of model constructs to document which netforms are best for communication and which are best for solution (not always the same); and (5) evolved computer implementation strategies to take fullest advantage of netform representations.

The fertile interaction between computer implementation technology and problem representation technology associated with these developments has greatly benefited from research on pure and generalized network problems, subsequently adapted to broader contexts.

Netform models are found in applications of scheduling, routing, resource allocation, production, inventory management, facilities location, distribution planning, among other areas. These new modeling techniques are mathematically and symbolically linked to network and augmented network structures that allow users to conceptualize formulations of their problems graphically. This pictorial aspect is extremely valuable in communicating and refining problem interrelationships without the use of mathematics and computer jargon. Thus, it protects the nontechnical person against technical legerdemain and exaggerated claims of model "realism." This technology also often yields a model that can be solved as a sequence of linear network problems or by merging solutions to linear networks in progressively refined stages.

It's useful to segregate netform models into several categories, starting with the most basic pure network models.

Pure Network Models

Pure network problems embody a group of distinct model types, including shortest path, assignment, transportation, and transshipment problems, the latter being the most general.

The transshipment model appears in many applications, either directly or as a subproblem. An illustration of the structure of a pure network transshipment model is given in Figures 1 and 2, which show different parts of the same problem. This model identifies the form of an investment and allocation problem that can be adapted to a variety of practical applications. The arrows in these network models are called *arcs* and the circles are called *nodes.*

Figure 1: **Investment & Allocation**

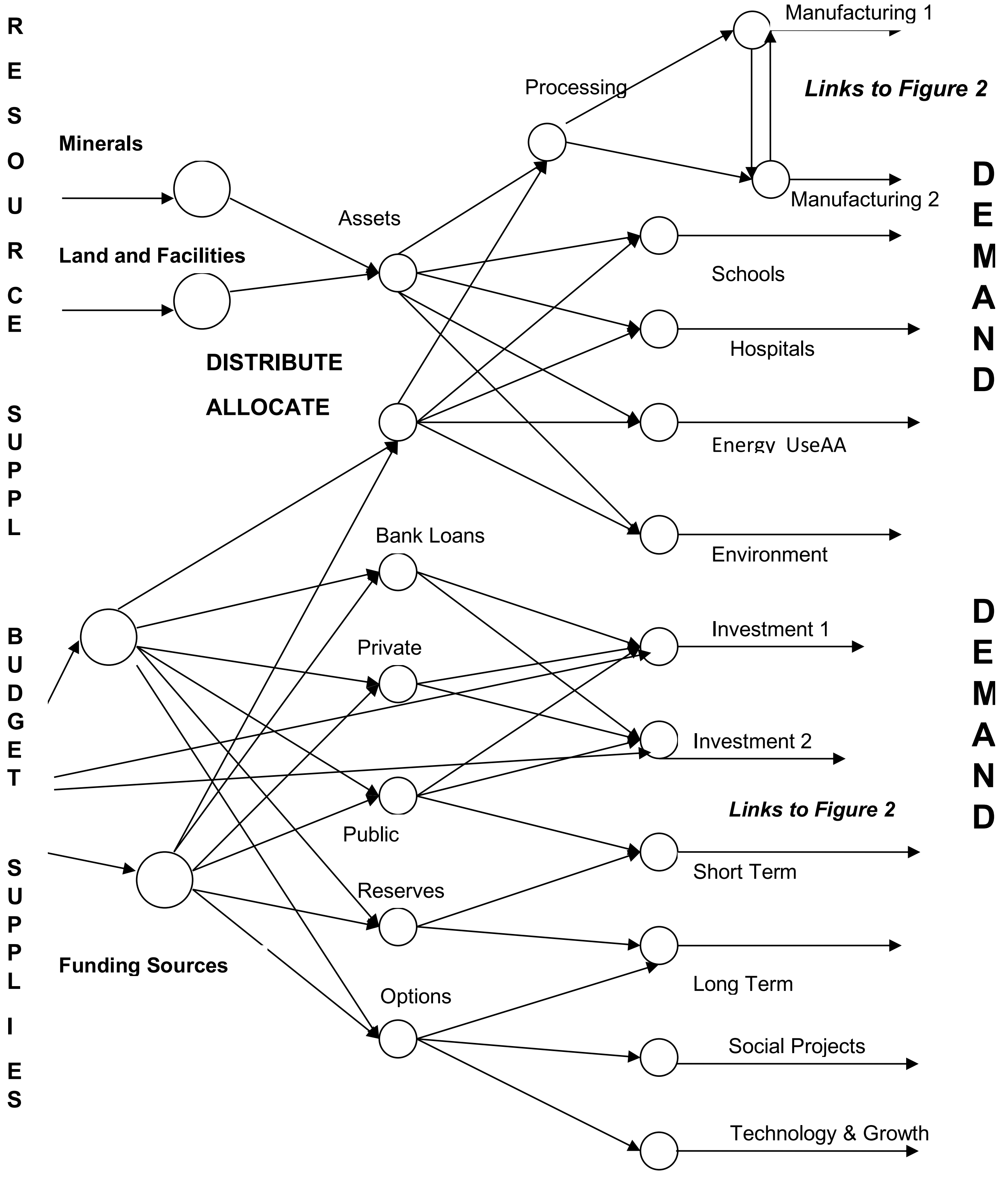

FIGURE 2: Investment & Allocation: Additional Potential Interactions

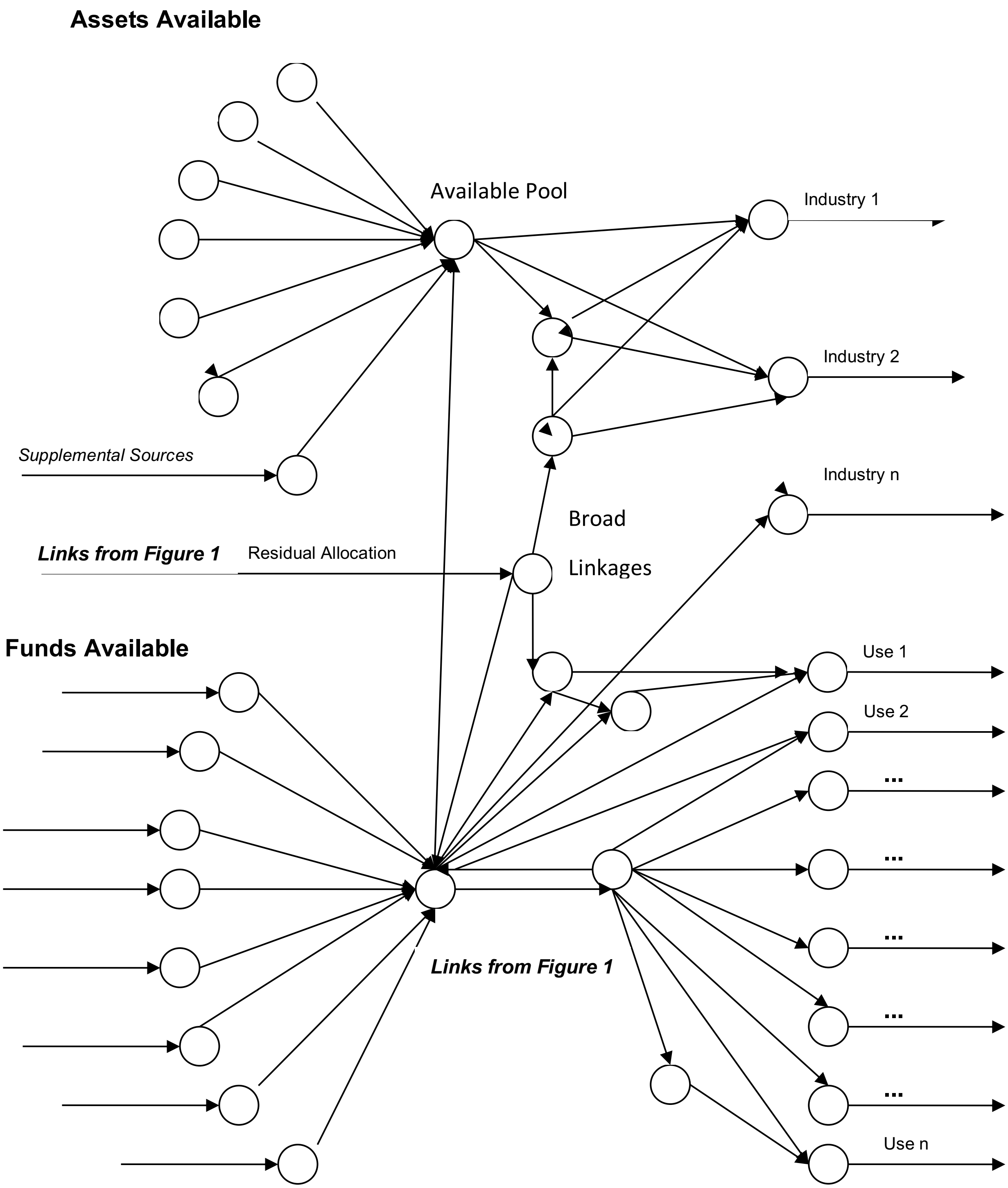

Figure 3 shows a pure network model in fuller detail with an illustration for a cash-flow problem. In this representation, nodes appear both at the beginning and the end of all arcs, so that arcs may be interpreted as allowable flow paths between nodes. We have also introduced triangles leading into or out of (some) nodes to represent *supplies* and *demands,* respectively. Arcs have lower and upper bounds on the flow permitted to cross them, shown in parentheses, and costs per each unit of flow, shown in rectangles.

The objective in the transshipment problem is to determine how much to ship along each arc within the limits stipulated by the bounds in order to satisfy all supplies and demands and to minimize total cost. Satisfying supplies and demands means that the total flow into the node minus the total flow out must equal the node's demand, and the total flow out of the node minus the total flow in must equal the node's supply. For all other nodes, the flow into the node must equal the flow out.

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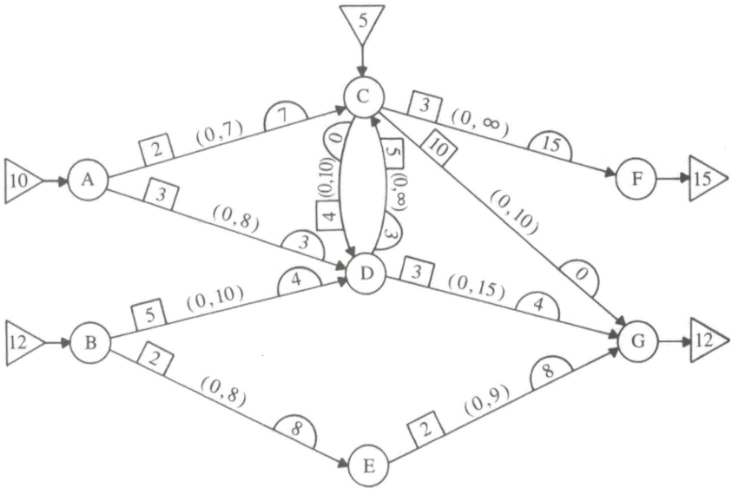
Figure 3: Capacitated transshipment cash-flow model



In this capacitated transshipment cash-flow model, the nodes correspond to subsidiaries of a central company that operates in different locations. The supplies and demands represent excess or deficit cash, respectively. Thus, nodes *A, B,* and C have excess funds, nodes *D* and £ have no funds, and nodes f and G have deficit funds.

The arc from node *A* to node C indicates that it is possible to transfer funds from subsidiary *A* to subsidiary C. The absence of an arc indicates that it is not possible to transfer funds directly between the corresponding pair of subsidiaries (though it may be possible to transfer funds indirectly by means of a sequence of arcs through intermediate subsidiaries). The arc from node *A* to node C has a lower bound of 0, an upper bound of 7, and a cost of 2. The numbers in the semi-circles on the arcs illustrate a solution which satisfies the *node equations* and the bound requirements for the mathematical formulation of the problem given in the Appendix.

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A significant practical feature of these models is that advanced network flow optimization algorithms can solve problems containing millions of variables (arcs) within seconds to minutes, even without parallel implementation or the use of GPUs.

Applications of Pure Network Problems

Pure network problems provide models for many mathematical optimization problems and for major components of many additional problems. Inventory maintenance problems for example, typically exhibit an underlying network structure. A cousin of the inventory maintenance problem is the PERT/CPM problem, which seeks the best way to sequence a complex set of interdependent activities. The PERT/ CPM framework, which constitutes one of the simplest network model forms, has been used in a variety of practical applications (including construction of the Polaris submarine) and has been reported to save substantial costs and greatly speed completion of complex projects.

Other types of problems involving the effective management of resources also frequently exhibit network structures. Such problems are becoming increasingly important in government and industry. Direct network formulations of water resource management problems, for example, are finding widespread use where canals, river reaches, and pipelines take the role of arcs, while reservoirs and pumping stations take the role of nodes. Planning over time frequently looms as a major consideration in these applications.

The Texas Water Development Board and the government of Poland have introduced what-if analyses into water resource management by using a succession of simulations having alternative supply and demand configurations and solving the resulting network for each simulation run. The step of finding an optimal solution to each network problem is used to determine the best response to meet demands for water use, given a specified supply-and-demand configuration. (This use of simulation, in which parameters are varied to achieve what-if analyses by means of rigorous solution techniques, is to be contrasted with the common "quasi-optimization" use of simulation.) To analyze the full range of relevant configurations, roughly 500 such runs must be made each month. The feasibility and cost-effectiveness of such runs is due to the efficiency with which the underlying networks are solved.

The problem of determining flows and heads in a general pipeline system (such as in municipal water systems) with reservoirs, pumps, gate and check valves, given fixed inputs and withdrawals is equivalent to a convex transshipment problem under the assumption of convex head losses. Such problems are easily solved as ordinary transshipment problems using a piecewise linear approximation of the convex function. Since the convexity requirements are usually satisfied for real pipe networks, this is an example of another class of real-world problems that can now be handled by network procedures with far greater effectiveness than by the procedures used in the past.

Another important instance of the use of network models occurs in manpower promotion and assignment problems. Such models can be used to guarantee acceptable hiring and promotion policies in accordance with government rules and regulations. A variety of cash-management problems can also be modeled as transshipment problems. These models include sources of funds in addition to cash (such as maturing accounts and notes receivable, sales of securities, and borrowing) and uses of funds other than a single investment. The generalized network model discussed later makes it possible to further incorporate discount, interest, and other financial considerations directly into the model.

Many nonlinear problems involve network subproblems. One of the most basic and prevalent forms of nonlinear problems is the fixed-charge network problem, whose major offshoots include the genre known as location problems. The nonlinear element of a fixed-charge network is the fixed-charge arc, which has the following special property: whenever the arc is "used" (that is, permitted to transmit flow), a charge is incurred that is independent of the amount of flow across the arc. Fixed-charge networks have been used to model problems of plant and warehouse location, equipment purchasing and leasing, personnel hiring, and offshore oil-drilling platform location, among others.

To better utilize forecasting data and to support economically rational operational decisions, an optimization-based decision support system for planning supply, distribution, and marketing, integrates key economic and physical supply, distribution, and marketing characteristics over a short-term planning horizon that incorporates inventory planning and time lags for manufacturing and distribution. To model this timing problem, the model is partitioned into time zones and employs replications of the basic model to accommodate the distinct time periods (e.g., weeks), as shown in Figure 4.

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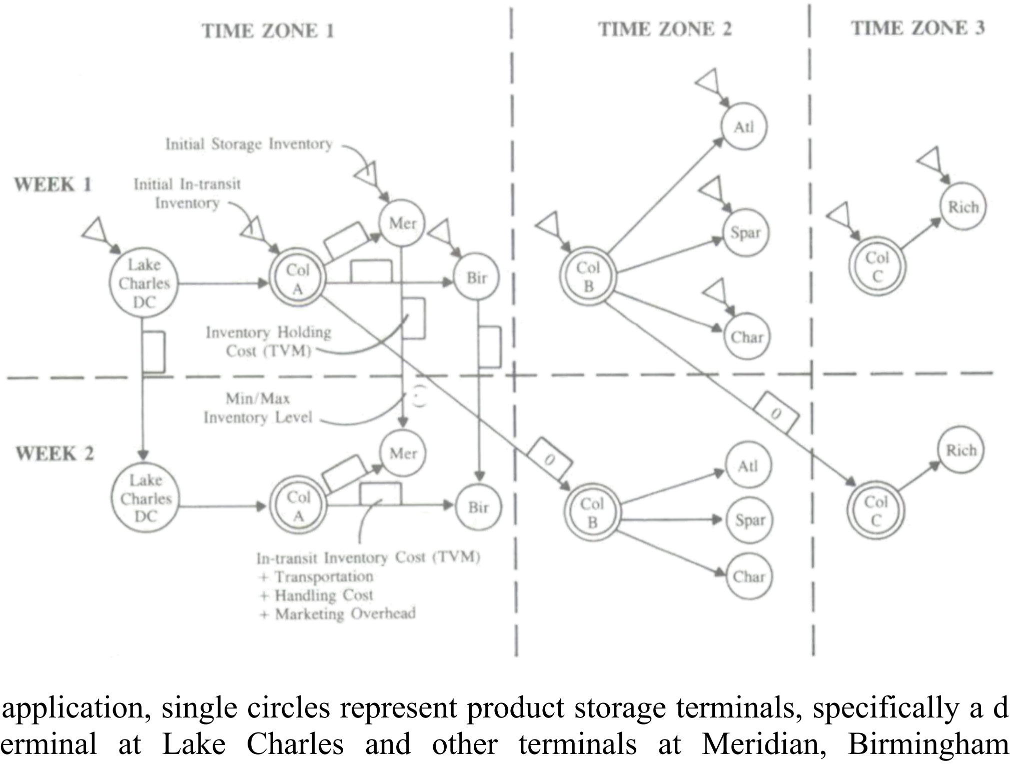
Figure 4: Optimal inventory for storage terminals across time zones and periods



For this application, single circles represent product storage terminals, specifically a distribution center terminal at Lake Charles and other terminals at Meridian, Birmingham, Atlanta, Spartanburg, Charlotte, and Richmond. Double circles represent locations in a distribution pipeline called Colonial in each time zone. For example, "Col A" represents the segment of the Colonial pipeline in time zone 1 which contains the product storage terminals that are within one week's travel time from the refinery. "Col B" represents the segment of the Colonial pipeline in time zone 2 which contains those terminals between one and two weeks away, etc. (Double circles are used for pipeline nodes to make the network easier to read. Due to the large size of the problem and the number of different components which nodes were used to represent, the network diagram for the entire problem also employs additional node symbols, such as half circles and ellipses.) The vertical arcs be-tween product storage terminal nodes represent inventory held at storage terminals between time periods. The diagonal arcs between pipeline nodes represent in-transit inventory. For example, product which is in the *"A"* segment of the Colonial pipeline in week 1 will travel to the *"B"* segment in week 2 (if it is not lifted into the Meridian or Birmingham terminals). Thus week 1 demand in lime zones 2 and 3 can only be satisfied by initial in-transit or storage inventory, modeled by the supply triangles shown, since product cannot travel from Lake Charles to Col B or Col C in one week. (Demand triangles, not shown, would be attached to each product terminal node.)

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Top management can use such a system to make decisions as where to sell products, what price to charge, where to buy or trade products, how much to buy or trade, how much of a product to hold in inventory, and how much product to ship by each mode of transportation. All information can be summarized by location, by line of business, and by week, incorporating the critical timing considerations associated with all these decisions.

The network framework as shown in Figure 4 is important for two main reasons. First, the pictorial aspect of network models is valuable for working with a broad spectrum of personnel to develop, validate, and implement an integrated modeling approach. Second, network models permit rapid solution speed, which allows management to respond quickly to the dynamics of a commodity market industry by what-if sessions.

Another application of pure network transshipment modeling involves making personnel assignment decisions. A variety of goals may be relevant for assigning personnel to positions and each goal may have several sub-goals, leading to a model that has many criteria or objective functions. These criteria express the desire to fill as many positions as possible in some priority fashion, the cost of assigning a person to a position, the utility of such an assignment to the organization, and the desirability of the assignment to the person assigned. In addition, the organization may have special priorities for filling particular positions and for handling shortages within a job type. The speed of solving network models makes it possible to solve the problem with different weighting strategies for aggregating the decision criteria and for evaluating the implications of these strategies.

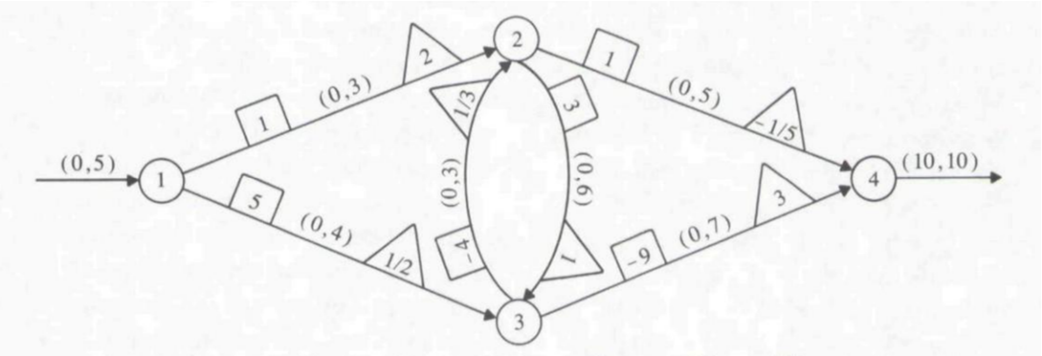
Generalized Networks

The generalized network (GN) problem represents a class of LP problems that has received much less attention than it deserves. Generalized networks include pure networks as a special case, and the additional applications made available by these models rival or even surpass the applications of pure networks in their practical significance. GN problems arise in resource allocation, production, distribution, scheduling, capital budgeting, and other settings.

An important distinction exists between arcs in pure network problems and arcs in GN problems. An arc of a generalized network has a *multiplier.* (In pure networks, this multiplier is always equal to 1.) This distinction is illustrated by the GN problem shown in Figure 5 along with the associated network. As with pure network problems, a node corresponds to a problem equation and an arc corresponds to a problem variable. Consequently, each arc has a cost, lower bound, and upper bound. Costs are shown within rectangles, and bounds are shown within parentheses. Arc multipliers are shown within triangles.

The multiplier of a generalized network problem acts upon the flow across an arc; the amount of flow starting out on the arc will not necessarily be the amount arriving at the opposite end. Specifically, the flow entering the arc is multiplied by the value of the multiplier to produce the quantity of flow leaving the arc. The arc's cost, lower bound, and upper bound refer only to the units of flow entering the arc.

Figure 5: Generalized Network



Minimize $1X_{12} + 5X_{13} + 3X_{23} + 1X_{24} - 4X_{32} - 9X_{34}$

$$\begin{array}{cccccccccc}
X_1 & -X_{12} & -1X_{13} & & & & & & = & 0 \\
 & 2X_{12} & & -1X_{23} & -1X_{24} & +1/3X_{32} & & & = & 0 \\
 & & 1/2X_{13} & +1X_{23} & & +1X_{32} & -1X_{34} & & = & 0 \\
 & & & & -1/5X_{24} & & +3X_{34} & -X_4 & = & 0
\end{array}$$

$0 \leq X_1 \leq 5;$ $0 \leq X_{12} \leq 3;$ $0 \leq X_{13} \leq 4;$ $0 \leq X_{23} \leq 6;$

$0 \leq X_{24} \leq 5;$ $0 \leq X_{32} \leq 3;$ $0 \leq X_{34} \leq 7;$ $0 \leq X_4 \leq 10;$

In this generalized network representation, supply into node 1 and demand out of node 4 are modeled using bounded arcs, rather than supply and demand triangles. Thus node 1 has a supply of at most 5 while node 4 has a demand of exactly 10. (The arc out of node 4 could have been modeled alternatively as a demand triangle with a demand of 10. In that case the variable X4 would be omitted, and the last equation would be $-1/5X_{24} + 3X_{34}, = 10$.) If 2 units start on the arc from node 1 to node 2, the multiplier of 2 will cause 4 units to arrive al node 2. Likewise, 10 units starting on the arc from node 2 to node 4 will result in -1 units arriving at node 4 (or, equivalently, 2 units leaving node 4 on that arc) since the multiplier in this case is -1/5.

Applications of Generalized Networks

Generalized networks can successfully model many problems that have no pure network equivalent. This is made possible by two useful interpretations of arc multipliers. First, multipliers can be viewed as modifying the amount of flow of a specified item. By means of flow modification, generalized networks can model such situations as evaporation, seepage, deterioration, breeding, interest rates, sewage treatment, purification processes with varying efficiencies, machine efficiency, and structural strength design. However, it is also possible to interpret the multiplication process as transforming one type of item into another. This interpretation provides a way to model such processes as manufacturing, production, fuel to energy conversion, blending, crew scheduling, manpower to job requirements, and currency exchanges. The following applications are examples of possible uses of generalized networks.

A water distribution system with losses can be modeled as a generalized network problem, as in a situation where water is moved through canals to various reservoirs. However, such a model can also consider the retention of water over several time periods. The multipliers in this case represent the losses from evaporation and seepage.

File reduction problems can be expressed in the form of a generalized network model with a single extra constraint. The aim is to facilitate the reduction of extremely large microdata files to smaller, statistically representative files, using an objective function of minimizing the amount of information lost in the reduction process. The arcs represent paths from the original records to the reduced records. A nonzero flow on an arc implies that the original record is to be represented by the reduced record connected to it by that arc. The multipliers on the arcs are used to ensure that the reduced file is truly representative of all the original records.

Another illustrative application of generalized networks consists of representing copper refining processes where the electrolytic refining procedure is carried out by a large dc electrical network. The arcs are current paths with the multipliers representing the appropriate resistances. The model makes it possible to identify the effect of short circuits in the refining process.

Generalized networks can likewise be used for warehouse funds-flow optimization. The warehouse funds-flow model is a multi-time period model where the arcs are used to represent sales, production, and inventory of both products and cash. The multipliers are introduced to facilitate the conversions between cash and products.

Multi-national cash management problems are natural candidates for analysis using generalized networks, which are capable of incorporating transfer pricing, receivables and payables, collections, dividend payments, interest payments, royalties, and management fees. The arcs represent possible cash-flow patterns and their costs and savings, while the multipliers capture liquidity changes and exchange rates.

Two early applications that can now be carried out more effectively involve a model for distributing natural gas across the U.S. and a model for managing public rangelands. While the Alaskan Pipeline was being built, Congress funded the development of a model for natural gas distribution that included the largest 240 distributors and the largest 100 pipelines in the United States. Multiple objective functions were used to represent the hierarchical way gas is distributed (residential customers before commercial customers, and so forth). A generalized network was used to model the problem, accounting for the fact that gas in the pipeline is used to drive the pipeline pumps, and multipliers on arcs capture the condition of losing gas as it moves along the pipeline.

The model for determining the optimal management of public rangelands was used to determine uses of land for grazing and for supporting wildlife. This required identifying optimal numbers of animals of various types, in consideration of their dietary requirements, that would be allowed to populate specific regions. The large size of the problem arose from the massive area of rangelands involved and the multiple nutritional and geographic factors affecting the possible allocations of different wildlife types to these regions. A simplified component of the model is shown in Figure 6. Until altered political priorities turned attention to different applications, the resulting netform-based system was routinely used an average of 700 times a month throughout the US.

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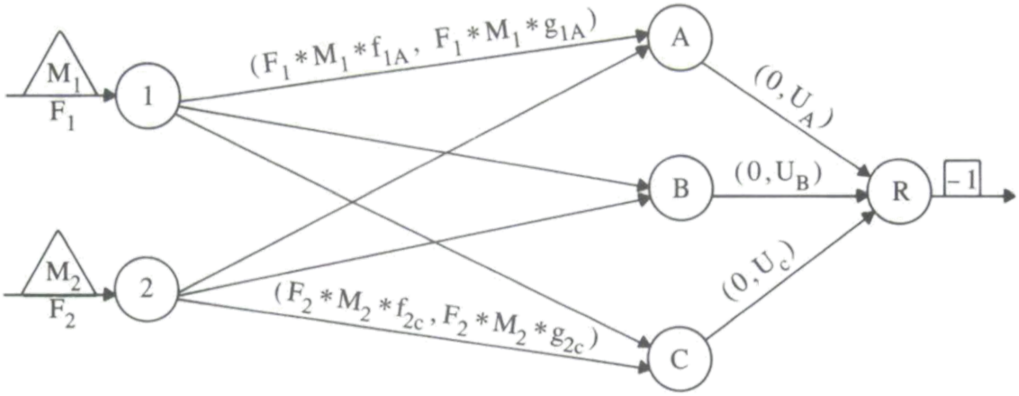
Figure 6: US Bureau of Land Management model



In the geographical region represented by this diagram, there are two types of animals, represented by the nodes labeled 1 and 2, each of which can consume varying amounts of three types of plants, represented by the nodes labeled A, B and C. The volume of food consumed by each animal of types 1 and 2 are represented by the multipliers M_1 and M_2, respectively, shown associated with the arcs entering nodes 1 and 2. These multipliers transform numbers of animals flowing on these arcs, denoted by F_1 and F_2 and determined by allocations in other parts of the network, into total volume of food consumed. Each of the arcs from the animal nodes to the plant nodes has a lower and upper bound of the special form illustrated within parentheses for the first and last of these arcs. The distinguishing feature of these bounds is that they are variable, and depend on the amounts of flow F_1 and F_2 entering nodes 1 and 2. Specifically, two fractions f_{1A} and g_{1A} are applicable to the arc from node 1 to node A, indicating that the amount of flow on this arc must be at least $F_1*M_1*f_{1A}$ and at most $F_1*M_1*g_{1A}$ (which identifies the least and greatest quantities of plant A that type 1 animals will consume in an appropriate diet). Similar variable bounds bracket the flows on the other arcs from animal nodes to plant nodes, as illustrated on the arc from node 2 to node C. The total available quantity of each of the three plant types in the given region is represented by the bounds on the arcs out of nodes A, B and C. The BLM objective was to utilize the total plant food available to the fullest extent possible, subject to the dietary requirements of the various animal species. This was accomplished by placing a cost of - 1 (that is, a profit of 1) on the arc leaving node R and using a cost-minimizing algorithm to solve the problem.

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Additional applications of generalized networks include machine loading problems, blending problems, and scheduling problems such as production and distribution problems, crew scheduling, aircraft scheduling, and manpower training.

Integer Generalized Networks

An especially important realm of application extends the use of generalized networks by requiring that flows on certain generalized arcs must occur in integer (whole number) amounts. Introducing the integer requirement into the GN problem enables it to model an unexpected diversity of additional applications, including such problems as scheduling variable-length television commercials into time slots, assigning jobs to computers in computer networks, scheduling payments on accounts where contractual agreements specify lump sum payments, and designing communication networks with capacity constraints.

Using integer requirements in GN problems enables any 0-1 LP problem to be modeled as an integer GN problem, although models with a natural GN structure are most advantageously treated in this way. Integer generalized networks are particularly useful to model requirements that flows on certain arcs must equal their upper or lower bounds, or that at least (or at most) a specified number of arcs from particular sets must have zero flows and other similar logical requirements. These netform representations consisting of networks with easily specified side conditions can rigorously express all problem elements that would ordinarily require expression in an algebraic form (as by customary mathematical programming formulation techniques). Consequently, they effectively replace the obscure and unilluminating algebraic representation by an equivalent, but much easier to understand, pictorial representation.

Figure 7 illustrates a useful modeling device based on integer constrained generalized arcs commonly employed in the netform approach.

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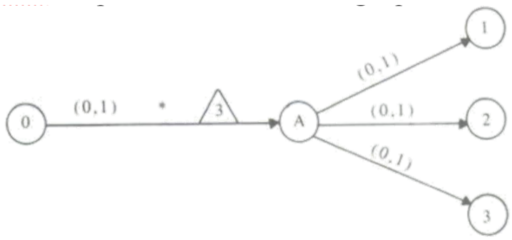
Figure 7: Model component for a generalized network with integer flows



This figure represents only a component of a model, and thus no supplies or demands are shown. The bounds, costs, and multipliers are depicted by the same conventions employed before. In addition, the asterisk on the arc from node 0 to node A indicates that its flow must be an integer. If the flow is 0, then 3-0 = 0 and no flow gets transmitted to node A. But if the flow is 1, then 3 units are transmitted to node A. Further, be-cause of the upper bounds of 1 on each of the three arcs leaving node A, the only possible way to distribute the three units flowing into node A is to send exactly one unit to each of the nodes 1, 2, and 3. Thus, by giving all arcs bounds of 0 and 1 and introducing a generalized arc, the following effect has been achieved: when the flow on the arc from node 0 to node A is 0, the flow on each of the three arcs out of node A is 0; when the flow on the arc from node 0 to A is 1, the flow on each of the three arcs out of node A is 1.

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An extension of this device to handle an even more useful set of conditions is illustrated in the model component shown in Figure 8. The combination of arc multipliers and 0-1 integer restrictions gives rise to what generally is called an integer network or a 0-1 generalized network.

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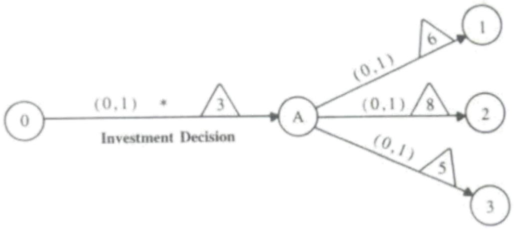
Figure 8: Model component for equipment investment



This figure is the same as Figure 7 except that multipliers have now been added to the three arcs leaving node A. For concreteness, we may suppose this diagram represents an investment decision: to invest in project A (if the flow on the arc from 0 to A is 1) or not to invest in project A (if the flow on the arc from 0 to A is 0). Then the nodes 1, 2, and 3 identify different components of this investment project. In this example, the investment project is set in an equipment purchase context, and nodes 1, 2, and 3 represent different equipment types. In other contexts, these nodes might represent different types of aircraft in a fleet, different parcels of land in a real estate venture, different types of stock in a portfolio, etc. The multipliers on the arcs into nodes 1, 2, and 3 represent the quantities of each component of project A (each type of equipment) that would be acquired if the decision is made to invest in that project. By the conventions and flow relationships previously described, the diagram of Figure 8 transforms the investment into its components in precisely the manner desired; that is, a flow of 1 on the arc from node 0 to node A (representing the decision to invest) translates into six units of equipment 1 at node 1, eight units of equipment 2 at node 2, and five units of equipment 3 at node 3.

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Future Directions

We are entering a new age of computer applications. The evolution of ideas and perspectives that has led to expert systems, knowledge engineering, object oriented programming, and intelligent work stations has underscored the fundamental role of representational systems in interactions between human beings and machines. Because we tend to express complex relationships by pictures and diagrams, technology is relying increasingly on pictorial representations as an indispensable element of new advances.

Netforms are being drawn into this process in a natural fashion and can be employed to improve the design of intelligent interfaces on several fronts. Netforms' legacy of successful applications of visual modeling technology, spanning diverse business, government and scientific settings, provides a background for bringing pictorial representations into human-machine interfaces to achieve practical, bottom-line benefits.

With these innovative developments, netforms are influencing how problems are translated into solvable forms and how they are conceptualized and communicated. More important, these applications are affecting the range of problems that are perceived as susceptible to formulation, enlarging the domains that can be represented for obtaining improved insights and solutions.

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Appendix: Mathematical Statement of a Transshipment Network Problem

The description of how a transshipment problem may be stated mathematically shows the connections between graphical and algebraic structures and applies with natural extensions to model other classes of network and netform problems. To state a network problem in algebraic form, we define a variable for each arc. For example, let X_{ij} denote the flow on the arc from node i to node j and c_{ij} denote the unit cost on this arc. An arc is denoted as an ordered pair (i,j) where the first component specifies the *from node* and the other component the *to node.* Next, we create the objective function for the problem as an expression involving the costs and variables. For the problem in Figure 3, the objective function would be

$$2X_{AC} + 3X_{AD} + 5X_{BD} + 2X_{BE} + 4X_{CD} + 3X_{CF} + 10X_{CG} + 5X_{DC} + 3X_{DG} + 2X_{EG}$$

Upon identifying the objective function, we create a constraint for each node that expresses the restriction on flow into and out of the node. To do this, it is convenient to view a supply as an inflow and a demand as an outflow. Then the requirement at each node can be expressed as Total Inflow - Total Outflow, or equivalently. Total Inflow - Total Outflow = 0.

The customary transposition of any constant term of this equation to the right band side causes supplies to be denoted as negative quantities and demands to be denoted as positive quantities. To see this, consider constructing the "Total Inflow - Total Outflow = 0" equation for node C in Figure 3. The total flow into node C consists of the five units of supply and $(X_{AC} + X_{DC})$- The total flow out of node C is $X_{CF} + X_{CG} + X_{CD}$- Thus we have $5 + X_{AC} + X_{DC} – (X_{CF} + X_{CG} + X_{CD}) = 0$, or after transposing the constant term, $X_{AC} + X_{DC} - X_{CF} - X_{CG} - X_{CD} = - 5$. Thus, the supply of five becomes expressed as a negative quantity because of its movement to the right band side of the equality sign. On the other hand, a demand, (which is included in the outflow that is subtracted in the "in-flow-outflow" equation), appears as a positive quantity when moved to the right-hand side. This also discloses, incidentally, that a supply may be viewed as a negative demand, and vice versa. The entire algebraic statement of the capacitated transshipment cash flow problem shown in Figure 3 is as follows.

Minimize $2X_{AC} + 3X_{AD} + 5X_{BD} + 2X_{BE} + 4X_{CD} + 3X_{CF} + 10X_{CG} + 5X_{DC} + 3X_{DG} + 2X_{EG}$

Subject to

$$
\begin{array}{rrrrrrrrrrcr}
-X_{AC} & -X_{AD} & & & & & & & & & = & -10 \\
 & & -X_{BD} & -X_{BE} & & & & & & & = & -12 \\
X_{AC} & & & & -X_{CD} & -X_{CF} & -X_{CG} & +X_{DC} & & & = & -5 \\
 & X_{AD} & +X_{BD} & & +X_{CD} & & & -X_{DC} & -X_{DG} & & = & 0 \\
 & & & X_{BE} & & & & & & -X_{EG} & = & 0 \\
 & & & & & X_{CF} & & & & & = & 15 \\
 & & & & & & X_{CG} & & +X_{DG} & +X_{EG} & = & 12
\end{array}
$$

$0 \leq X_{AC} \leq 7$; $0 \leq X_{AD} \leq 8$; $0 \leq X_{BD} \leq 10$; $0 \leq X_{BE} \leq 8$;

$0 \leq X_{CD} \leq 10$; $0 \leq X_{DC}$; $0 \leq X_{CF}$; $0 \leq X_{CG} \leq 10$;

$0 \leq X_{DG} \leq 15$; $0 \leq X_{EG} \leq 9$;

Each X_{ij} appears in exactly two node equations, (the equation for node / and the equation for node j). Further, since X_{ij} contributes to the outflow of the node i equation and to the inflow of the node j equation, it appears with a coefficient of -1 in the former and with a coefficient of +1 in the latter. Thus, each column of the coefficient matrix has one -1 and one +1 entry. If the restrictions at the nodes are stated as inequalities rather than equations (that is, Total Inflow - Total Outflow < 0 or Total Inflow - Total Outflow > 0), then slack or surplus variables are added to convert these constraints to equations, and it is admissible for columns to contain a single nonzero entry. Inequality restrictions at nodes generally arise by stipulating that supplies or demands must be "at least" or "at most" a certain amount.

By reversing the above steps, it is possible to represent any LP problem whose coefficient matrix has at most one +1 and at most one - 1 entry per column as a transshipment problem. That is, to construct a graph that corresponds to such a problem, create a node for each constraint and an arc for each variable, affixing supplies, demands and bounds in the manner indicated.