A class of multi-level balanced Foundation-Penalty cuts for mixed-integer programs

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Abstract: Glover and Sherali (2003) introduced a wide class of Foundation-Penalty (FP) cuts for GUB and ordinary mixed-integer programs. The FP cuts are easy to generate by exploiting routine branch-and-bound penalty calculations, and encompass other classical cuts such as disjunctive cuts, lift-and-project cuts, convexity cuts, Gomory cuts, and mixed-integer rounding cuts. Here we focus on two special classes of FP cuts, called *balanced FP* cuts and *multi-level* balanced *FP* cuts, and exhibit their relationship to special forms of disjunctive cuts. A numerical example illustrates the rich variety of cuts that can be generated.

Keywords: mixed-integer programming; foundation-penalty cuts; optimisation.

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1 Introduction

Consider a mixed-integer program MIP stated in the form

$$\textbf{MIP}: \text{Minimise } c^T x \quad \text{subject to } x \in X \equiv X_1 \cap X_2, \tag{1}$$

where $x \in R^n$, X_1 describes a set of constraints representing a polyhedron in R^n, $X_2 \equiv \{x : x_j$ is integer-valued for $j \in I \subseteq N\}$, and where $N = \{1, ..., n\}$ is the index set of all the variables.

This paper complements the work of Glover and Sherali (2003) in which a new class of cutting planes for Mixed-Integer Programs, called *Foundation-Penalty* (FP) cuts, was introduced. As the name suggests, FP cuts are predicated on two elements: a (linear) foundation function, and a set of penalties that are computed based on the conditional values taken on by either a single integer variable, which might be restricted to be binary-valued, or by several binary variables comprising a *Generalised Upper Bounding* (GUB) set. The concept underlying the derivation of FP cuts was shown to bear a relationship to the various classes of disjunctive cuts (Balas, 1998; Balas et al., 1993; Sherali and Adams, 1990, 1994), convexity cuts (Glover, 1973, 1975), Gomory cuts (Gomory, 1960a, 1960b), and mixed-integer rounding cuts (Marchand and Wolsey, 2001).

In this paper, we focus on an important special class of FP cuts, namely, *balanced FP* cuts, which are derived by constructing a foundation function in a manner that effectively yields equal values of the penalty on alternative branching decisions or disjunctive statements. By involving additional suitable constraints in combination with a focal source row for which an initial FP cut is generated, we derive a class of *multi-level balanced FP* cuts. These cuts have the flexibility of producing a rich variety of alternative valid inequalities that can serve to tighten the representation of the mixed-integer program along selected dimensions.

The remainder of this paper is organised as follows. The next section provides the basic background material concerning the derivation of FP cuts. Section 3 then introduces the concept of balanced FP cuts and exhibits its relationship to disjunctive cuts. Multi-level extensions of these cuts that consider focal as well as secondary constraints in the cut generation process are discussed in Section 4. Section 5 presents an illustrative example to demonstrate the rich variety of cuts that are afforded by this class of multi-level FP cuts, and Section 6 concludes the paper with recommendations for further extensions and research in this context.

2 Basic Foundation-Penalty (FP) cuts

As mentioned in Section 1, the class of FP cuts is governed by two principal elements, namely, a foundation function, and certain penalty computations conditioned on values taken on by either a single integer/binary variable or by a set of GUB-constrained binary variables.

The foundation function is some selected linear function of the form $\sum_{j\in J} d_j x_j$, where $J \subseteq N$. Typically, this function might correspond to a reduced cost objective representation associated with some dual feasible solution, or more pertinently, an optimal basis to the Linear Programming (LP) relaxation $\overline{\text{MIP}}$ of MIP, given by

$$\mathbf{MIP} : \text{Minimise } \{c^T x : x \in \overline{X}\}, \tag{2}$$

where $\overline{X}$ denotes the usual LP relaxation of *X:* In this case, we would have

$$J \equiv \{\text{set of nonbasic variablesg}\}, \text{ and } d_j \geq 0, \forall j \in J. \tag{3}$$

The penalty computations are conducted with respect to some integer/binary variable x_k, $k \in I$, or with respect to a set of binary variables that are GUB-constrained according to

$$\sum_{k\in K} x_k = 1, \quad \text{where } K \subseteq I. \tag{4}$$

Suppose that the foundation function conforms with equation (3) corresponding to the LP relaxation solution, and that an integer-restricted variable x_k currently takes on a value b_k that is fractional. Let P_k^+ and P_k^- be the respective values of penalties derived via the following augmented LP relaxations:

$$P_k^+ = \min\left\{ \sum_{j\in J} d_j x_j : x \in \overline{X}, x_k \geq \lfloor b_k \rfloor + 1 \right\}, \tag{5a}$$

$$P_k^- = \min\left\{ \sum_{j\in J} d_j x_j : x \in \overline{X},\ x_k \leq \lfloor b_k \rfloor \right\}. \tag{5b}$$

Then, Glover and Sherali (2003) show that the following inequality defines a valid FP cut:

$$\sum_{j\in J} d_j x_j \geq P_k^+ \left(x_k - \lfloor b_k \rfloor\right) + P_k^- \left(\lfloor b_k \rfloor + 1 - x_k\right). \tag{6}$$

Moreover, (6) is a separating inequality if either $P_k^+ > 0$ or $P_k^- > 0$. For the special case of a binary variable x_k, $k \in I$, we define

$$P_{k1} = \min\left\{ \sum_{j\in J} d_j x_j : x \in \overline{X},\ \text{and } x_k = 1 \right\} \text{ and} \tag{7a}$$

$$P_{k0} = \min\left\{ \sum_{j\in J} d_j x_j : x \in \overline{X}, \quad \text{and } x_k = 0 \right\}. \tag{7b}$$

More generally, P_{k1} and P_{k0} could be any lower bounds on the respective values of MIP under the corresponding additional conditions based on the disjunction that $x_k = 1$ or $x_k = 0$, respectively. For example, these values could be based on the simple penalties derived via a single dual simplex pivot on an optimal LP tableau for $\overline{\text{MIP}}$ that has been augmented by the additional restriction $x_k = 1$ or $x_k = 0$, or via multiple dual simplex pivots of this type as used in strong branching strategies (see Balas, 1979). Alternatively, we could solve integer knapsack relaxations based on surrogate constraint strategies (see Rardin and Karwan, 1984). Of course, if any of the penalty computations yield $P_{k1} = \infty$ or $P_{k0} = \infty$, we simply enforce the opposite restriction $x_k = 0$ or $x_k = 1$, respectively, and conduct subsequent implied reductions via standard logical tests (see Nemhauser and Wolsey, 1999). Hence, in what follows, we will always assume that all penalties derived are finite.

Furthermore, in the binary GUB case, we compute P_{k1} as above for each $k \in K$, where for any $k \in K$, the computation of P_{k1} in (7a) is conducted by also explicitly enforcing $x_j = 0$, $\forall j \in K - \{k\}$ by virtue of the presence of equation (4) within the defining set *X*, and similarly, by setting to zero variables belonging to other GUB sets that contain x_k. Then, Glover and Sherali (2003) show that FP cut for the case of a single binary variable x_k is given by

$$\sum_{j\in J} d_j x_j \geq P_{k1}x_k + P_{k0}(1-x_k), \tag{8a}$$

and for the GUB-constrained case (4), the FP cut is given by

$$\sum_{j\in J} d_j x_j \geq \sum_{k\in K} P_{k1}x_k. \tag{8b}$$

Moreover, in either case equations (8a) or (8b), under the condition (3) corresponding to an optimal basis for the LP relaxation $\overline{\text{MIP}}$ of MIP, if any of the penalties are positive for a currently fractional variable x_k in the LP solution, then equation (8a) provides a separating inequality that deletes this LP solution.

3 Balanced FP cuts

In this section, we introduce a special class of FP cuts, called *balanced FP* cuts, which are derived with respect to a foundation function $\sum_{j\in J} d_j x_j$ such that the penalties on the right-hand-sides of equation (6) or equations (8(a), (b)) are equal to some common value $P > 0$. In this case, setting $P_k^+ = P_k^- = P$ in equation (6) or $P_{k1} = P_{k0} = P$ in equation (8a) or $P_{k1} = P, \forall k \in K,$ in equation (8b) we obtain the *balanced FP* cut given by

$$\sum_{j\in J} d_j x_j \geq P. \tag{9}$$

It is instructive to see how this balanced FP cut arises in the context of a disjunctive cut (see Balas, 1979, 1998; Glover, 1975; Sherali and Shetty, 1980). Toward this end, we focus on the general integer variable case embodied by the FP cut (equation (6)), which deals with a single branching variable for an MIP problem. Similar constructs are possible for single or GUBconstrained binary variables addressed by the FP cuts equations 8(a) and 8(b). Denote an LP representation of the MIP problem relative to a current basis by the matrix equation

$$x_I + Ax_J = b,$$

where x_I and x_J respectively represent the vectors of the basic and non-basic variables, and x_J consists of the variables x_j, $j \in J$. Under the assumption that the basic variables are required to be nonnegative, we have the associated inequality

$$Ax_J \leq b. \tag{10}$$

Suppose that the branching variable x_k has a current basis representation given by $x_k + A_k x_J = b_k$. Then, writing $x_k = b_k - A_k x_J$ and imposing the branching or disjunctive inequalities $x_k \geq \lfloor b_k \rfloor + 1$ or $x_k \geq \lfloor b_k \rfloor$ in concert with any additional inequalities from equation (10), we obtain the disjunction, say,

$$\{A^+ x_J \leq b^+\} \text{ or } \{A^- x_J \leq b^-\}. \tag{11}$$

Selecting non-negative weight vectors w^+ and w^- for surrogating the two respective sets of constraints in equation (11) yields the implied disjunction

$$\{w^+A^+x_J \leq w^+b^+\} \text{ or } \{w^-A^-x_J \leq w^-b^-\}, \tag{12}$$

where w^+ and w^- are required to be such that w^+b^+ and w^-b^- are both negative, thereby assuring that these surrogate constraints will be violated by the basic solution that sets $x_J = 0$. Furthermore, by way of normalisation, we elect to produce a common right-hand-side value in these surrogate constraints, given by $w^+b^+ = w^-b^- = -P$, for a chosen value $P > 0$. Define

$$z^+ = -w^+A^+ \text{ and } z^- = -w^-A^-. \tag{13}$$

Then by the indicated normalisation, the disjunction equation (12) becomes, using equation (13),

$$\{z^+x_J \geq P\} \text{ or } \{z^-x_J \geq P\}. \tag{14}$$

Consequently, by the disjunctive cut principle (Balas, 1979; Glover, 1975; Sherali and Shetty, 1980), we derive the valid inequality

$$dx_J \geq P, \tag{15}$$

where

$$d_j \equiv \max\{z_j^+, z_j^-\} \quad \forall j \in J. \tag{16}$$

Observe that equation (15) is a balanced FP cut, predicated on the foundation function $\sum_{j\in J} d_j x_j$ whose coefficients are given by equation (16). These coefficients can, in fact, be derived via a separation-type LP of the following form, which attempts to strengthen the cut coefficients in equation (15) along desired selected dimensions by appropriately choosing a coefficient vector $q \geq 0$, $q \neq 0$.

$$\text{Minimise} \left\{ \begin{aligned} & q^T d : d \geq -w^+A^+, \\ & \quad d \geq -w^-A^-, \\ & \quad -w^+b^+ = -w^-b^- = P, \\ & \quad (w^+, \quad w^-) \geq 0 \end{aligned} \right\}, \tag{17}$$

where $P > 0$ is a selected constant (e.g., $P = 1$).

Remark 1: Observe that we could commence a process of this type using a foundation function defined by, say, the reduced costs as in Balas, 1998), then solve equations (5a) and (5b) to generate nonnegative dual variables to be used as the surrogation weights w^+ and w^- in order to derive a cut (equation (15)) based on the revised consequent values of d_j, $\forall j \in J$, as given by equation (16). In this process, we are not restricted to use the same initial foundation function in equations (5a) and (5b). Instead, we could use different initial functions, called *sub-foundation functions*, for deriving penalties and dual or surrogate multipliers for each branch. The resultant z^+ and z^- values given by equation (13) can then be used to derive the final foundation function as per equation (16), to yield the balanced FP cut (equation (15)).

Remark 2: A similar surrogation process based on the disjunction $x_k = 1$ or $x_k = 0$ for the case of a single binary variable can be used to produce the corresponding vectors z^1 and z^0, analogous to equation (13), yielding equation (15) via the resulting disjunction equation (14), where

$$d_j \equiv \max\{z_j^1, z_j^0\}, \quad \forall j \in J.$$

Likewise, for the GUB-constrained case embodied by equation (4), the disjunction

$$\bigvee_{k \in K} \{x_k = 1\}$$

leads to the analogous surrogate coefficients vectors z^k, $k \in K$, yielding the balanced FP cut (equation (15)) with

$$d_j \equiv \max\{z_j^k, k \in K\}, \quad \forall j \in J.$$

Remark 3: Suppose that two integer-restricted variables x_h and x_k currently take on fractional values b_h and b_k, respectively. Let $P_{hk}^{++}, P_{hk}^{+-}, P_{hk}^{-+}, and\ P_{hk}^{--}$ be the respective penalty values of the LP relaxations derived relative to the four branching conditions

$$\left\{x_h \geq \lfloor b_h \rfloor + 1 \text{ and } x_k \geq \lfloor b_k \rfloor + 1\right\},$$
$$\left\{x_h \geq \lfloor b_h \rfloor + 1 \text{ and } x_k \leq \lfloor b_k \rfloor\right\},$$
$$\left\{x_h \leq \lfloor bh \rfloor \text{ and } x_k \geq \lfloor bk \rfloor + 1\right\},$$
$$\text{and } \left\{x_h \leq \lfloor b_h \rfloor \text{ and } x_k \leq \lfloor b_k \rfloor\right\}.$$

Assume that each of these values is positive and let $z_j^{++}, z_j^{+-}, z_j^{-+}$, and z_j^{--}, $\forall j \in J$, denote the corresponding surrogate coefficient values obtained by using the optimal dual variables to surrogate the respective LP relaxation constraints, so that the corresponding right-hand-sides of these surrogate constraints are the respective aforementioned penalty values. Then a valid balanced FP cut

$$\sum_{j \in J} d_j x_j \geq P$$

is produced by scaling these surrogate constraints so that all right-hand-sides equal the same positive penalty P, and then by the disjunctive cut principle, selecting d_j to be the maximum coefficient of x_j over these scaled surrogate constraints, $\forall j \in J$. The extension to simultaneously treat three or more branching disjunctions is apparent.

4 Multi-level balanced FP cuts for general MIP problems

In this section, we describe extensions to the basic balanced FP cuts discussed in Section 3 to derive a class of multi-level balanced FP cuts. For the sake of exposition, we will focus on the branching disjunction $x_k \geq \lfloor b_k \rfloor + 1$ or $x_k \leq \lfloor b_k \rfloor$ for a general integer-restricted variable in MIP, where b_k is the current fractional value taken on by x_k in the solution to $\overline{\text{MIP}}$, with similar extensions to the other cases discussed in the previous section being evident.

Suppose that the current basis representation for the branching variable x_k is given by

$$x_k + \sum_{j \in J} a'_{kj} x_j = b_k. \tag{18}$$

We wish to impose the disjunction that

$$x_k \geq \lfloor b_k \rfloor + 1 \equiv b_k + (1 - f_k)$$

or

$$x_k \leq \lfloor bk \rfloor \equiv b_k - f_k,$$

where $0 < f_k < 1$ is the fractional part of b_k. We could apply this disjunction directly to equation (18), or alternatively, in order to derive typically stronger cuts, we can adopt the usual practice (see Nemhauser and Wolsey, 1999) of writing

$$a'_{kj} = \lfloor a'_{kj} \rfloor + f_{kj}, \forall j \in J_1$$
$$\equiv \{j \in J : x_j \text{ is integer-restricted and } 0 \leq f_{kj} \leq f_k\},$$

and

$$a'_{kj} = \lceil a'_{kj} \rceil - (1 - f_{kj}),\ \forall j \in J_2$$
$$\equiv \left\{j \in J : x_j \text{ is integer-restricted and } f_k < f_{kj} < 1\right\}.$$

Then, we can impose that the integral quantity

$$x_k + \sum_{j \in J1} \lfloor a_{kj} \rfloor x_j + \sum_{j \in J2} \lceil a_{kj} \rceil x_j,$$

which is currently equal to b_k for the given basic solution, should likewise be either greater than or equal to $\lfloor b_k \rfloor + 1$ or less than or equal to $\lfloor b_k \rfloor$. This leads to the disjunction

$$\left\{-\sum_{j \in J} a_{kj} x_j \geq (1 - f_k)\right\} \quad \text{or} \quad \left\{\sum_{j \in J} a_{kj} x_j \geq f_k\right\}, \tag{19}$$

where

$$\begin{aligned} a_{kj} &= f_{kj}, && \forall j \in J_1, \\ a_{kj} &= -(1 - f_{kj}), && \forall j \in J_2, \quad \text{and} \\ a_{kj} &= a'_{kj}, && \text{otherwise}. \end{aligned} \tag{20a}$$

Alternatively, without such a decomposition of the coefficients of the integer-restricted variables, we would obtain the disjunction equation (19) with

$$a_{kj} = a'_{kj}, \quad \forall j \in J. \tag{20b}$$

4.1 First-level cuts

Using weights f_k and $(1 - f_k)$ for the respective disjunctive statements in equation (19), we can re-write the resulting scaled disjunction as

$$\left\{\sum_{j\in J} z_j^+ x_j \geq P\right\} \quad \text{or} \quad \left\{\sum_{j\in J} z_j^- x_j \geq P\right\}, \tag{21a}$$

where,

$$\begin{aligned} &z_j^+ \equiv -f_k a_{kj}, \quad \forall j \in J, \\ &z_j^- \equiv (1-f_k)a_{kj}, \quad \forall j \in J, \quad \text{and } P \equiv f_k(1-f_k). \end{aligned} \tag{21b}$$

By the disjunctive cut principle, this yields a level-one balanced FP cut of the form

$$\sum_{j\in J} d_j^1 x_j \geq P, \text{ where } d_j^1 \equiv \max\{z_j^+, z_j^-\}, \quad \forall j \in J. \tag{22}$$

Note from equation (21b) that

$$\begin{aligned} &d_j^1 = z_j^+ = -f_k a_{kj} \quad \text{if} \quad a_{kj} \leq 0 \quad \text{and} \\ &d_j^1 = z_j^- = (1-f_k)a_{kj} \quad \text{if} \quad a_{kj} > 0. \end{aligned} \tag{23}$$

This resulting *first-level balanced FP* cut corresponds to the MIP cut of Gomory (1960a, 1960b).

4.2 Second-level cuts

Observe that the first-level cut described above can be viewed as one that is derived by considering the problem to minimise $\sum_{j\in J} d_j x_j$ subject to the constraints of $\overline{\text{MIP}}$ in addition to either one of the cuts that describe the disjunction (21a), and then performing a single dual simplex pivot on this cut row to derive the penalty *P* in either case. This leads to the valid cut (22).

More advanced balanced FP cuts that offer a greater variety of structures can be generated via penalties obtained from two or more dual pivots. In a special case involving a two-pivot penalty determination, we show that it is possible to identify an explicit formula for the *z*-values in the resulting equivalent disjunction, and therefore facilitate the quick generation of cuts for this case. We focus in particular on the situation where one of the branch penalties is obtained from two dual pivots and the other penalty is obtained from a single dual pivot, and represent the resulting *second-level balanced FP* cut by

$$\sum_{j\in J} d_j^2 x_j \geq P.$$

To characterise this cut, denote the branching equation for launching two dual pivots by

$$\sum_{j\in J} (-a_j)x_j \geq -b_0. \tag{24}$$

We call this the *focal* branch equation and call the equation for the other branch the *alternative* branch equation. Note that we can write the focal and alternative branch equations as respectively given by the disjunction

$$\left\{\sum_{j\in J} z_j^* x_j \geq P\right\} \quad \text{or} \quad \left\{\sum_{j\in J} z_j^{\#} x_j \geq P\right\}, \tag{25a}$$

where by equation (24), we have (noting that $P > 0$ and $b_0 < 0$)

$$z_j^* \equiv \frac{Pa_j}{b_0}, \quad \forall j \in J. \tag{25b}$$

Now, denote the basic equation chosen as a source for the second dual pivot by

$$x_i + \sum_{j\in J} a_{ij} x_j = bi, \quad \text{or} \quad \sum_{j\in J} (-a_{ij})x_j \geq -b_i. \tag{26}$$

This second equation must yield an infeasible (negative) value for x_i after pivoting on the focal branching equation, and hence there must exist some $p \in J$ such that $a_p < 0$ and

$$b_i - (a_{ip} / a_p) b_0 < 0.$$

By implication, if x_i begins feasible in the current basic solution, then $a_{ip} > 0$. Based on equations (23)–(25), we can now formulate the valid disjunction that

$$\left\{\begin{aligned} &\sum_{j\in J} (-a_j)x_j \geq -b_0 \\ &\sum_{j\in J} (-a_{ij})x_j \geq -b_i \end{aligned}\right\} \quad \text{or} \quad \left\{\sum_{j\in J} z_j^{\#} x_j \geq P\right\}. \tag{27}$$

Let w^* and w_i denote the respective dual variables or nonnegative surrogate multipliers attached to the two inequalities on the left in equation (27) to obtain the equivalent statement

$$\left\{\sum_{j\in J} [-w^* a_j - w_i a_{ij}] x_j \geq P\right\} \quad \text{or} \quad \left\{\sum_{j\in J} z_j^{\#} x_j \geq P\right\}, \tag{28a}$$

where $w^* \geq 0$ and $w_i \geq 0$ are such that

$$-b_0 w^* - w_i b_i = P, \quad \text{i.e.,} \quad w^* = -\frac{[w_i b_i + P]}{b_0}. \tag{28b}$$

Using equation (28b) to eliminate w^* from equation (28a), and substituting equation (25b), this reduces the disjunction equation (28a) to

$$\left\{\sum_{j\in J} [z_j^* - w_i v_j] x_j \geq P\right\} \quad \text{or} \quad \left\{\sum_{j\in J} z_j^{\#} x_j \geq P\right\}, \tag{29a}$$

where

$$v_j \equiv a_{ij} - \left(\frac{b_i}{b_0}\right) a_j, \quad \forall j \in J. \tag{29b}$$

The second-level balanced FP cut derived by applying the disjunctive cut principle to equation (29a) is then given by

$$\sum_{j \in J} d_j^2 x_j \ge P, \text{ where } d_j^2 \equiv \max\{z_j^* - w_i v_j, z_j^\#\}, \ \forall j \in J. \tag{30}$$

Further, in order to compare the first-level and the second-level cuts, let us define

$$J(v^+) = \{j \in J : v_j > 0\} \text{ and } J(v^-) = \{j \in J : v_j < 0\},$$
$$J(a^+) = \{j \in J : a_j > 0\} \text{ and } J(a^-) = \{j \in J : a_j < 0\}. \tag{31}$$

We now examine different *breakpoint values* w_{ij} for w_i, as governed by each possible $j \in J$ for which the two values in the maximand in equation (30) become equal. Equating these two terms when $v_j \neq 0$ gives

$$w_i = w_{ij} \equiv (z_j^* - z_j^\#)/v_j = [(Pa_j/b_0) - z_j^\#] = v_j.$$

Note that since w_i should additionally be nonnegative, and that $z_j^\#$ and a_j, $j \in J$, have the same sign since the coefficients of the focal and alternative branching equations in equation (25a) are oppositely signed, the relevant cases for computing these breakpoints are when $v_j < 0$ and $a_j > 0$, or when $v_j > 0$ and $a_j < 0$. That is, the breakpoints of interest are given by

$$w_{ij} = \frac{z_j^* - z_j^\#}{v_j} = \frac{(Pa_j/b_0) - z_j^\#}{v_j},$$
$$\forall j \in J_{bp} \equiv \{j \in J : v_j a_j < 0\}. \tag{32}$$

To compare the second-level cut equation (30) with the first-level cut, which is essentially given via the disjunction (25a) as

$$\sum_{j \in J} d_j^1 x_j \ge P, \text{ where } d_j^1 \equiv \max\{z_j^*, z_j^\#\}, \quad \forall j \in J, \tag{33}$$

we can directly compare the coefficients d_j^1 vs. d_j^2 for different cases as expounded by the following result.

Theorem 1: *For $w_i > 0$, the second-level cut is:*

- *deeper than the first-level cut $(d_j^2 < d_j^1)$ relative to all $j \in J(v^+) \cap J(a^-)$ and for any given such j attains its maximum depth for $w_i = w_{ij}$;*
- *shallower than the first-level cut $(d_j^2 > d_j^1)$ relative to all $j \in J(v^-) \cap J(a^+)$ such that $w_i > w_{ij}$, and to all $j \in J(v^-) \cap (J - J(a^+))$;*
- *equal in depth to the first-level cut $(d_j^2 = d_j^1)$ otherwise.*

In particular, the cut for

$$w_i = w_{i^*} \equiv \max\{w_{ij} : j \in J(v^+) \cap J(a^-)\}$$

dominates all cuts for $w_i > w_{i^*}$.

Proof: Consider $j \in J(v^+) \cap J(a^-)$, as addressed in Case (a). Note from equation (25b) that since $a_j < 0$, we have $z_j^* > 0$ (because $P > 0$ and $b_0 < 0$), and $z_j^\# < 0$ (because $z_j^\#$ has the opposite sign as z_j^*, being the alternative branch in equation (25a)). Hence, from equation (33), we get that $d_j^1 = z_j^*$, and from equation (30), we get

$$d_j^2 = z_j^* - w_i v_j < d_j^1 \quad \text{for } 0 < w_i < w_{ij}, \text{ and}$$
$$d_j^2 = z_j^\# < d_j^1 \quad \text{for } w_i \ge w_{ij}.$$

This establishes Case (a).

Next, consider $j \in J(v^-) \cap J(a^+)$. Hence, since $a_j > 0$, we get $z_j^* < 0$ and $z_j^\# > 0$ from equation (25). Therefore, $d_j^1 = z_j^\#$ from equation (33). Moreover, from equation (30), $d_j^2 = z_j^\#$ so long as $0 < w_i \le w_{ij}$ (this yields part of Case (c), therefore), but once $w_i > w_{ij}$ we get

$$d_j^2 = z_j^* - w_i v_j > z_j^\# = d_j^1,$$

thereby establishing the first part of Case (b). To see the second part, note that when $j \in J(v^-) \cap (J - J(a^+))$, we have $a_j \le 0$, yielding from equation (25) that $z_j^* \ge 0$ and $z_j^\# \le 0$. Hence, $d_j^1 = z_j^*$. Moreover, $z_j^* - w_i v_j > z_j^* \ge z_j^\#, \forall w_i > 0$, and so, $d_j^2 > d_j^1$ for all $w_i > 0$. This proves Case (b). Finally, examining the remaining case of $j \in J(v^+) \cap (J - J(a^-))$, we get $a_j \ge 0$, yielding via equation (25) that $z_j^* \le 0$ and $z_j^\# \ge 0$. Therefore, $d_j^1 = z_j^\# = d_j^2$ because $z_j^\# \ge z_j^* - w_i a_j, \forall w_i \ge 0$. This, together with the fact that $d_j^1 = d_j^2$ when $v_j = 0$ establishes Case (c) and completes the proof.

Theorem 1 discloses that the breakpoint values (32) give the interesting candidate values for w_i for generating second-level cuts, and that it is unnecessary to consider such values larger than the *critical breakpoint value* w_{i^*}.

The ability to select the parameter w_i to equal different breakpoint values provides the flexibility to produce trade-offs between the depth of the cut along different dimensions. We show how this occurs via a numerical example in the next section.

5 An illustrative example

Consider the following two equations, consisting of a source equation for branching and a second equation from a current primal feasible basic solution. The first five non-basic variables x_1 through x_5 and the basic variable x_7 are integer-restricted, while the non-basic variable x_6 and the basic variable x_8 are continuous.

$$x7 - 1.1x_1 + 2.8x_2 + 1.9x_3 - 3.7x_4$$
$$+ 2.4x_5 + 1.8x_6 = 3.6 \text{ (Source)} \tag{34a}$$

$$x_8 + 0.9x_1 + 1.6x_2 - 0.2x_3 - 1.3x_4 - 2.1x_5 + 2.3x_6 = 1.6. \quad (34b)$$

We first illustrate the derivation of a first-level balanced FP cut from the source equation. We begin by reducing the coefficients of the integer-restricted variables to their fractional parts according to equation (20a). Noting that $f_k = 0.6$, so that $J_1 = \{4,\ 5\}$ and $J_2 = \{1,\ 2,\ 3\}$, we get that the disjunction (19) is given by

$$0.1x_1 + 0.2x_2 + 0.1x_3 - 0.3x_4 - 0.4x_5 - 1.8x_6 \geq 0.4 \text{ (UpBranch)} \quad (35a)$$

or

$$-0.1x_1 - 0.2x_2 - 0.1x_3 + 0.3x_4 + 0.4x_5 + 1.8x_6 \geq 0.6 \text{ (DownBranch)}. \quad (35b)$$

We now apply weights $10f_k = 6$ and $10(1 - f_k) = 4$ to equations (35a) and (35b), respectively, to obtain equation (21a) as follows, where we have arbitrarily used a multiplier of ten for numerical convenience:

$$0.6x_1 + 1.2x_2 + 0.6x_3 - 1.8x_4 - 2.4x_5 - 10.8x_6 \geq 2.4 \text{ (giving } z_j^+) \quad (36a)$$

$$-0.4x_1 - 0.8x_2 - 0.4x_3 + 1.2x_4 + 1.6x_5 + 7.2x_6 \geq 2.4 \text{ (giving } z_j^-). \quad (36b)$$

Setting $d_j^1 \equiv \max\{z_j^+, z_j^-\}, \forall j \in J,$ as in equation (22), we obtain

$$0.6x_1 + 1.2x_2 + 0.6x_3 + 1.2x_4 + 1.6x_5 + 7.2x_6 \text{ (Foundation function)} \quad (37a)$$

$$0.6x_1 + 1.2x_2 + 0.6x_3 + 1.2x_4 + 1.6x_5 + 7.2x_6 \geq 2.4 \text{ (First-level balanced FP cut)}. \quad (37b)$$

The penalty $P = 2.4$ may be verified to result by performing a single dual pivot on each of the two branch equations (independently) relative to the objective of minimising the foundation function. (As previously noted, this first-level cut corresponds to the Gomory MIP cut).

Next, we illustrate an associated second-level balanced FP cut by selecting equation (35a) as the focal branch equation, (35b) as the alternative branch equation, and equation (34b) as the row (26) selected for augmenting the focal equation to launch a second-level pivot. Noting that the z_j^* and $z_j^{\#}$ values are given respective by z_j^+ and z_j^- as defined in equations (36a) and (36b), $\forall j \in J,$ the disjunction (27) is of the form:

$$\left\{\begin{array}{l} 0.1x_1 + 0.2x_2 + 0.1x_3 - 0.3x_4 - 0.4x_5 \\ \quad - 1.8x_6 \geq 0.4 - 0.9x_1 - 1.6x_2 + 0.2x_3 \\ \quad + 1.3x_4 + 2.1x_5 - 2.3x_6 \geq -1.6 \end{array}\right\}$$

or

$$\left\{\begin{array}{l} -0.4x_1 - 0.8x_2 - 0.4x_3 + 1.2x_4 \\ \quad + 1.6x_5 + 7.2x_6 \geq 2.4 \end{array}\right\}. \quad (38)$$

This leads to the following disjunction of the form equation (29a), where we have retained $w_i \equiv w_8$ as a variable:

$$\left\{\begin{array}{l} (0.6 - 0.5w_8)x_1 + (1.2 - 0.8w_8)x_2 + (0.6 + 0.6w_8)x_3 \\ \quad + (-1.8 + 0.1w_8)x_4 + (-2.4 + 0.5w_8)x_5 \\ \quad + (-10.8 - 9.5w_8)x_6 \geq 2.4 \end{array}\right\}$$

or

$$\left\{0.4x_1 - 0.8x_2 - 0.4x_3 + 1.2x_4 + 1.6x_5 + 7.2x_6 \geq 2.4\right\}. \quad (39)$$

The set J_{bp} in equation (32) is given by $J_{bp} = \{1,\ 2,\ 4,\ 5\}$, and the corresponding breakpoints w_{8j} for $j \in J_{bp}$, are given via equation (32) as follows:

$$w_{81} = 2, \quad w_{82} = 2.5, \quad w_{84} = 30, \quad w_{85} = 8. \quad (40)$$

Noting that $J(v^+) = \{1,\ 2,\ 6\}$, and $J(a^-) = \{1,\ 2,\ 3\}$, we get $J(v^+) \cap J(a^-) = \{1,\ 2\}$, and the critical breakpoint value w_{8*} defined in Theorem 1 is given by $w_{82} = 2.5$, and hence, the breakpoint values of interest are $w_{81} = 2$ and $w_{82} = 2.5$. Setting w_8 equal to each of these two values in turn, and generating the coefficients d_j^2 as specified by equation (30), gives the two second-level balanced FP cuts shown below in equations (41b) and (41c), respectively. The first-level cut equation (37b), is again displayed in equation (41a) below as a basis of comparison. (Note that the corresponding foundation functions, by construction, are simply the left-hand-sides of these cut equations.

$$0.6x_1 + 1.2x_2 + 0.6x_3 + 1.2x_4 + 1.6x_5 + 7.2x_6 \geq 2.4 \text{ (1st level)} \quad (41a)$$

$$-0.4x_1 - 0.4x_2 + 1.8x_3 + 1.2x_4 + 1.6x_5 + 7.2x_6 \geq 2.4 \text{ (2nd level: } w_8 = 2) \quad (41b)$$

$$-0.4x_1 - 0.8x_2 + 2.1x_3 + 1.2x_4 + 1.6x_5 + 7.2x_6 \geq 2.4 \text{ (2nd level: } w_8 = 2.5). \quad (41c)$$

Here, as w_8 takes successively larger values, the cut coefficients d_j^2 for $j \in J(v^+) \cap J(a^-) = \left\{1, 2\right\}$ progressively improve until reaching their limiting values (achieved for the associated breakpoints), while the coefficients for

$$j \cap J(v^-) \cap (J - J(a^+)) = \{3, 4, 5\} \cap \{1, 2, 3\} = \{3\}$$

progressively worsen. Since no breakpoints in equation (40) for

$$j \in J(v^-) \cap J(a^+) = \{3, 4, 5\} \cap \{4, 5, 6\} = \{4, 5\}$$

were smaller than the critical breakpoint $w_{8*} = 2.5$, none of the remaining coefficients changed. Indeed, as propounded by Theorem 1, for $w_8 > w_{85} = 8$, the coefficient for x_5 would have increased (worsened) beyond 1.6, and for $w_8 > w_{84} = 30$, the coefficient for x_4 would have increased (worsened) beyond 1.2.

We observe that, in contrast to the first-level cut whose coefficients are all nonnegative, it is possible to have negative coefficients in the second-level cut (corresponding to negative d_j coefficients in the foundation function at this level).

The construction that yields this outcome can also be applied by taking the cut source equation in the role of the x_i – equation (26). Thus, whenever a pivot on the chosen branching equation will render the source equation infeasible, a new cutting plane can be obtained directly from this source. In fact, this occurs in the present example.

6 Conclusions

In this paper, we have presented a new class of multi-level balanced Foundation Penalty (FP) cuts, which are easy to generate and that can further strengthen the standard Gomory (first-level balanced FP cuts) for Mixed-Integer Programs. While we have provided an explicit derivation and an illustrative example for the second-level balanced FP cut, a straightforward extension of this approach can be used to generate higher-level balanced FP cuts that involve a larger number of pivots. The idea here would be to start with any foundation function (including the one that produces a single pivot cut), and then to increase the value of the d_j-coefficients in the foundation function as needed to permit each successive pivot to be non-degenerate. Upon reaching a chosen stopping point, or upon terminating by reaching full primal feasibility, the penalty for the branch examined can be rescaled to equal the common P-value. The final form of the FP cut may then be produced by reference to the associated formulated disjunction via the disjunctive cut principle.

In fact, by adopting the alternative disjunctive cut viewpoint as afforded by our discussion in this paper, there is a rich variety of such cuts that could be generated, based on

- considering appropriate additional constraints to add to both sides of the initial branching inequalities
- formulating disjunctions involving more than one variable in this fashion (see Remark 3)
- formulating disjunctions that are predicated on more general constraints or cuts rather than the simple dichotomous inequalities treated herein that consider the up- and down-rounding of a single fractionating integer-restricted variable.

We propose to investigate extensions of this type along with an extensive computational study for future research.

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