Technical Appendix
Information sales and strategic trading

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Abstract

In this technical appendix we extend the results in the paper "Information sales and strategic trading" (forthcoming, *Review of Financial Studies*). We study the problem of a monopolist selling information to a set of risk-averse traders. We first analytically reduce the seller's problem to a simple constrained optimization, allowing for arbitrary allocations of information. We also fully characterize the equilibria in the models of Kyle (1985) and Kyle (1989) under general signal structures. Finally, we provide details on the numerical solutions to the information sales problems presented in the paper.

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1 Introduction

This technical appendix is a complement to the paper "Information sales and strategic trading" (forthcoming, *Review of Financial Studies*). It provides a more explicit characterization of the equilibria studied in the paper, as well as more details on the numerical analysis. For completeness we recall the main elements of the model. There are two assets in the economy: a risk-less asset in perfectly elastic supply, and a risky asset with a random final payoff $X \in \mathbb{R}$ and variance normalized to 1. All random variables are normally distributed, uncorrelated, and have zero mean, unless otherwise stated. There is random noise trader demand Z for the risky asset, and we let σ_z^2 denote the variance of Z. We study equilibria both in a setting where agents can submit price-contingent orders, which we refer to as "limit-orders," and a setting where agents can only submit "market-orders," i.e. they cannot condition their trades on price. We use the references "market-" and "limit-orders" to refer to the Kyle (1985) and Kyle (1989) models respectively, following the original paper by Kyle (1989), as well as Brown and Zhang (1997) and Bernhardt and Taub (2006).

There are N agents with CARA preferences with risk aversion parameter r_i to whom the monopolist can sell her information. Thus, given a final payoff π_i, each agent i derives the expected utility $\mathbb{E}[u(\pi_i)] = \mathbb{E}[-\exp(-r_i\pi_i)]$. Letting θ_i denote the trading strategy of agent i, i.e. the number of shares of the risky asset that agent i acquires, and assuming zero-initial endowments, the final wealth for agent i is given by $\pi_i = \theta_i(X - P_x)$, where P_x denotes the price of the risky asset. After the information sales stage, agent i will receive a signal that generates a filtration at the trading stage which we denote by $\mathcal{F}_i$. In a slight abuse of notation, we let $\mathcal{F}_u$ denote the information possessed by the uninformed. We characterize the allocations of information at the trading stage as containing signals of the form $Y_i = X + \delta + \epsilon_i$, where at this point the ϵ_i's can be arbitrarily correlated, and δ is a common noise term (that can stem from noisy information that the monopolist possesses, or added noise that is correlated across traders). We denote agent i's trading strategy in the limit-orders model by two positive constants (β_i, γ_i), defined by $\theta_i = \beta_i Y_i - \gamma_i P_x$, for $i = 1, \ldots, m$; whereas we use $\theta_i = \beta_i Y_i$ in the market-orders model.

We assume the existence of a competitive market-maker who sets prices conditional on order flow. This is a standard assumption in the context of the Kyle (1985) market-orders model, and it is isomorphic to the assumption of a competitive fringe of uninformed investors in the Kyle (1989) framework in that implies weak-form efficiency, or $P_x = \mathbb{E}[X|\mathcal{F}_u]$. Under our conjectured trading strategies, prices will be of the form

$$P_x = \lambda \left(\sum_{i=1}^{m} \beta_i Y_i - Z \right);$$

for some $\lambda > 0$.[1]

In section 2 we characterize the monopolist's problem by presenting a simple expression for traders' ex-ante certainty equivalent of wealth. In section 3 we give a characterization of the equilibria in both the Kyle (1985) and Kyle (1989) models with arbitrary allocations of information among the traders. In section 4 we characterize the equilibria in the symmetric case in the Kyle (1985) and Kyle (1989) models with risk-averse traders. Section 5 discusses the details on the numerical procedures we use in the paper to solve for the optimal information sales. Section 6 contains the proofs of the Lemmas.

2 The monopolist's problem

We start by generalizing Proposition 1 from the paper, which characterizes traders' ex-ante certainty equivalent, thereby providing a simple expression for the monopolist's problem. At this point, all we assume is that each of N agents receives a normally distributed signal Y_i, for $i = 1, \ldots, N$, where the actual correlation structure among the Y_i's is completely arbitrary (we shall consider special cases in the sections that follow). We recall the certainty equivalent for an informed agent in this class of models is given by

$$\mathcal{U}_i = -\frac{1}{r_i} \log\left(-\mathbb{E}[u(W_i)]\right).$$

The next lemma gives a simple expression for the *ex-ante* certainty equivalent in terms of the expected *interim* certainty equivalent χ_i (i.e., the certainty equivalent at the trading stage), given by

$$\chi_i = \mathbb{E}\left[\pi_i | \mathcal{F}_i\right] - \frac{r_i}{2} \text{var}\left(\pi_i | \mathcal{F}_i\right).$$

Lemma 1. *The ex-ante certainty equivalent for agent i is given by*

$$\mathcal{U}_i = \frac{1}{2r_i} \log\left(1 + 2r_i \mathbb{E}[\chi_i]\right); \tag{1}$$

where

$$\mathbb{E}[\chi_i] = \text{var}(\eta_i) \frac{(\lambda_i + r_i \xi_i/2)}{(\lambda_i + r_i \xi_i)^2}; \tag{2}$$

where $\eta_i \equiv \mathbb{E}\left[X - P_x | \mathcal{F}_i\right]$, $\xi_i \equiv \text{var}[X - P | \mathcal{F}_i]$ and $\lambda_i \equiv dP_x/d\theta_i$.

The above expression for the expectation of the interim certainty equivalent χ_i depends

[1] We remark that in the market-orders model λ measures the price impact of a trader's order, i.e. $\lambda = dP_x/d\theta_i$. We also note that in the limit-orders model this is not the case: since informed investors can submit price-contingent strategies, they affect the residual supply curve of a given agent. In the limit-orders we will use $\lambda_i = dP_x/d\theta_i$ to denote the price impact of an informed agent's trade.

only on the variance of the conditional expectation of profits per unit of asset demand, $\mathrm{var}(\eta_i)$, the price impact parameter λ_i, and the conditional quality of the information, measured by ξ_i. Each of these three terms will be characterized in sections 3 and 4 as functions of the model's primitives, and, in particular, of the monopolist's choice variables. We remark that the ex-ante certainty equivalent is a simple concave transformation of the interim certainty equivalent $\mathbb{E}[\chi_i]$.

The monopolist's problem reduces to the choice of a set of normally distributed signals $\mathbf{Y} \equiv \{Y_i\}_{i=1}^N$ that she will distribute to the N agents prior to trading. Depending on the information that the monopolist possesses (i.e. whether she knows X or only a noisy signal of X), the monopolist will be able to choose among different signals' allocations. In the most general case the signals will belong to a linear space $\mathcal{Y}$ of joint normally distributed random variables. The monopolist's problem can be stated as

$$\max_{\mathbf{Y}\in\mathcal{Y}} \quad \sum_{i=1}^{N} \frac{1}{2r_i} \log\left(1 + 2r_i \mathrm{var}(\eta_i) \frac{(\lambda_i + r_i\xi_i/2)}{(\lambda_i + r_i\xi_i)^2}\right). \tag{3}$$

In the above optimization problem, the endogenous variables $\mathrm{var}(\eta_i), \xi_i$ and λ_i are determined by a set of equilibrium conditions which are specific to the type of market considered (market- versus limit-orders), and which also depend on the actual allocation of information chosen by the monopolist. The following two sections characterize explicitly this dependence.

3 General characterization of equilibria

In this section we characterize the equilibria in the Kyle (1985) and Kyle (1989) models with N informed agents, each whom observes a signal of the form $Y_i = X + \delta + \epsilon_i$, where $\boldsymbol{\epsilon} \equiv \{\epsilon_i\}_{i=1}^N \sim \mathcal{N}(0, \boldsymbol{\Sigma}_\epsilon)$. We further assume agent i has CARA preferences with risk-aversion coefficient r_i. Thus we extend the analysis in the literature to allow for heterogeneous risk-aversion, as well as arbitrary signal structures.

We start by analyzing the equilibrium in the Kyle (1989) model. We solve for linear equilibria in which agent's i trading strategy is of the form $\theta_i = \beta_i Y_i - \gamma_i P_x$. For notational convenience we define $\sigma_i^2 = \mathrm{var}(\epsilon_i)$, $\beta_q = \boldsymbol{\beta}^\top \boldsymbol{\Sigma}_\epsilon \boldsymbol{\beta}$, and $\beta_{qi} = \mathbf{1}_i^\top \boldsymbol{\Sigma}_\epsilon \boldsymbol{\beta}$, where $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^N$, and $\mathbf{1}_i$ is a vector with 1 in the ith element and zero elsewhere. Finally, we let $\gamma_s = \sum_{i=1}^N \gamma_i$.

Lemma 2. *The equilibrium at the trading stage of the limit-orders model is characterized by $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ that solve*

$$\beta_i = \frac{\alpha_i}{\lambda_i + r_i\xi_i}, \quad i = 1, \ldots, m; \tag{4}$$

$$\gamma_i = -\frac{\upsilon_i}{\lambda_i + r_i \xi_i}, \quad i = 1, \ldots, m; \tag{5}$$

where λ_i, ξ_i, α_i and υ_i are stated explicitly in the proof as functions of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in (33), (35), (37) and (38).

The characterization of the equilibria in (4) and (5) consists of a system of $2N$ non-linear equations for the equilibrium values of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. In the numerical analysis we present below, we shall consider special cases of the information structure, summarized by $\boldsymbol{\Sigma}_\epsilon$, which admit simpler characterizations.

The monopolist's problem reduces to the maximization of (3), where λ_i and ξ_i are given by (33), (35). In order to be more explicit, we first note that we can write

$$\text{var}(\eta_i) = \alpha_i^2(1 + \sigma_\delta^2 + \sigma_i^2) + \frac{\upsilon_i^2(\beta_s^2(1 + \sigma_\delta^2) + \beta_q + \sigma_z^2)}{(1 + \gamma_s)^2} + \frac{2\alpha_i \upsilon_i(\beta_s(1 + \sigma_\delta^2) + \beta_{qi})}{(1 + \gamma_s)}. \tag{6}$$

Thus, the monopolist's problem is to maximize (3) over her (possibly constrained) choice variable $\boldsymbol{\Sigma}_\epsilon$, as well as over the equilibrium parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, subject to the constraints (4) and (5).

The next lemma extends the analysis to the market-orders model of Kyle (1985).

Lemma 3. *The equilibrium at the trading stage of the market-orders model is characterized by $\boldsymbol{\beta}$ that satisfies*

$$\beta_i = \frac{1 - \lambda(\beta_{qi} + \beta_s(1 + \sigma_\delta^2))}{(\lambda + r_i \xi_i)(1 + \sigma_\delta^2 + \sigma_i^2)}; \quad i = 1, \ldots, m; \tag{7}$$

where λ and ξ_i are stated explicitly in the proof as a function of $\boldsymbol{\beta}$ in (39) and (42).

We remark that, due to the nature of market-orders, the system (7) that characterizes the equilibrium is simpler than the system (4)-(5). Since agents cannot condition their trades on order-flow (or price), the equilibrium is characterize by a system of N equations, rather than $2N$.

The monopolist's problem is to maximize (3), where $\lambda_i = \lambda$ is given by (39), ξ_i by (42), and

$$\text{var}(\eta_i) = \frac{\left(1 - \lambda\left(\beta_{qi} + \beta_s\left(\sigma_\delta^2 + 1\right)\right)\right)^2}{1 + \sigma_\delta^2 + \sigma_i^2}. \tag{8}$$

subject to the constraints given in (7).

4 Characterization of symmetric equilibria

The main results in the paper concern the case of symmetric allocations of conditionally independent information, i.e., when the monopolist sells to $m \leq N$ agents, each of whom receives a signal of the form $Y_i = X + \epsilon_i$, with $\epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2)$ independent and identically distributed. We note the slight abuse of the term "symmetric," since this allocation is asymmetric among the group of m agents who receive a signal, and the $N - m$ who remain uninformed. We let $s_\epsilon = 1/\sigma_\epsilon^2$ denote the precision of the signals informed agents receive. We further assume all agents have the same risk-aversion parameter r.

We characterize the symmetric equilibrium by the trading aggressiveness parameter $\beta_i = \beta$ for all $i = 1, \ldots, m$. We define the informational content parameter ψ by $\text{var}(X|P_x)^{-1} \equiv \tau_u = 1 + \psi y$, where $y \equiv m s_\epsilon$ is the aggregate precision sold by the monopolist. Furthermore, we define the conditional precision of payoffs and trading profits for an informed agent as $\tau_i \equiv \text{var}(X|\mathcal{F}_i)^{-1}$ and $\tau_\pi \equiv \text{var}(X - P_x|\mathcal{F}_i)^{-1}$ respectively. In the market-orders model $\mathcal{F}_i = \sigma(Y_i)$, whereas in the limit-orders model $\mathcal{F}_i = \sigma(Y_i, P_x)$, so that $\tau_i = \tau_\pi$ in the later case. Finally, we define the informational incidence parameter $\zeta \equiv dP/d\mathbb{E}[X|\mathcal{F}_i]$, and the price impact parameter $\lambda_i \equiv dP_x/d\theta_i$.

The next lemmas characterize the equilibrium in each of these models fixing m and s_ϵ. Following Kyle (1989), we describe the equilibrium in terms of the parameters ψ and ζ, rather than the actual price coefficients. Lemma 4 is covered in Kyle (1989), but the characterization in Lemma 5 is new.

Lemma 4. *The equilibrium at the trading stage of the limit-orders model is characterized by $\beta \in \mathbb{R}_+$ such that*

$$\kappa \sqrt{\frac{\psi}{m(1-\psi)s_\epsilon}} = m \frac{(1-2\zeta)}{(1-\zeta)} \frac{(1-\psi)}{(m-\psi)}; \tag{9}$$

where

$$\frac{\zeta}{\tau_i} = \frac{\psi}{\tau_u}; \tag{10}$$

$$\tau_i = 1 + s_\epsilon + \phi(m-1)s_\epsilon; \tag{11}$$

$$\tau_u = 1 + \psi y; \tag{12}$$

and where the endogenous variables ϕ, ζ, ψ are given explicitly in the proof as a function of β in (46), (49) and (53). The certainty equivalent of the informed speculator's profit is given by

$$\mathcal{U}_i = \frac{1}{2r} \log\left(1 + \left(\frac{\tau_i}{\tau_u} - 1\right) \frac{(1-2\zeta)}{(1-\zeta)^2}\right); \quad i = 1, \ldots, m. \tag{13}$$

Equilibrium condition (9) can be treated as a non-linear equation for the trading aggressiveness parameter, β, since both ψ and ζ can be expressed in terms of β, see (46), and (53) in the proof. The lemma's proof is standard, and gives explicitly the relationship between all other endogenous parameters and the equilibrium value for β.

The next lemma presents an analogous characterization of the equilibrium in the market-orders model of Kyle (1985).

Lemma 5. *The equilibrium at the trading stage of the market-orders model is characterized by $\beta \in \mathbb{R}_+$ such that*

$$\kappa\sqrt{\frac{\psi}{(1-\psi)ms_\epsilon}} = \frac{(1+\psi s_\epsilon)\tau_u(1-2\zeta)}{\tau_i\tau_u - s_\epsilon(1-\psi)^2}; \tag{14}$$

where

$$\frac{\zeta}{\tau_i} = \frac{\psi}{1+\psi s_\epsilon}; \tag{15}$$

$$\tau_i = 1 + s_\epsilon; \tag{16}$$

$$\tau_u = 1 + \psi m s_\epsilon \tag{17}$$

and the endogenous variables, ψ and ζ are given explicitly as a function of β in the proof. The certainty equivalent of the informed speculator's profit is given by

$$\mathcal{U}_i = \frac{1}{2r}\log\left(1 + \left(\frac{\tau_\pi}{\tau_u} - 1\right)\frac{(1-2\zeta)}{(1-\zeta)^2}\right); \quad i = 1, \ldots, m; \tag{18}$$

with

$$\tau_\pi = \frac{\tau_i\tau_u^2}{\tau_i\tau_u - s_\epsilon(1-\psi)^2}. \tag{19}$$

As in the previous lemma, equilibrium condition (14) can be treated as a non-linear equation for the trading aggressiveness parameter, β, since both ψ and ζ can be expressed in terms of β. We finish by remarking that the characterization of both Lemma 4 and Lemma 5 shows that, in the symmetric case under consideration, the monopolist's problem depends on the risk-aversion parameter r and the noise-trading intensity σ_z only via their product $\kappa = r\sigma_z$.

5 Numerical algorithms

We consider the solution to the problem for different restrictions on the type of signals the monopolist can use. Throughout we will assume that all agents have the same risk aversion parameter, i.e. $r_i = r$ for all i. We first study the symmetric case discussed in section 4. We then turn to the cases where the monopolist gets a noisy signal.

Base case. We start with the simplest case, where the monopolist gets to choose the number of agents m to whom she sells her information, and the variance of the signals' error σ_ϵ^2 that she gives to the traders, where the signals' errors are independent. The problem for the limit-orders case then reduces to a two-dimensional maximization problem, over both m and σ_ϵ^2, subject to the equilibrium constraint. Since the equilibrium constraint is only given implicitly, in Lemma 4, in order to solve the problem, the monopolist must maximizes (3) over m, σ_ϵ^2, and one of the equilibrium variables (say β), such that (9) holds. We note that all variables in (3) and (9) can be expressed in terms of β, m and σ_ϵ^2 using the expressions in the proof of Lemma 4. In order to solve the problem numerically, and due to the discreteness in m, we solve for the optimal σ_ϵ^2 fixing m, and search over a large set of values for m. This way we simply have to face a two-dimensional optimization problem, over σ_ϵ^2 and β, with a single non-linear constraint, which can be dealt with using standard techniques.

Table 1 presents a set of numerical results for different values of the risk-aversion parameter r, fixing the volatility of noise traders at $\sigma_z = 1$, for the limit-orders market. When the monopolist sells to one agent our algorithm shows that she adds no noise to the signals (see the paper for an analytical proof). When risk-aversion is $r = 0.1$, 0.5, or 1, the monopolist optimally sells to a single agent. As soon as risk-aversion is above the threshold $\kappa^* \approx 1.74$, the monopolist optimally sells to as many agents as she can signals with vanishing precision.[2]

The optimization problem in the market-orders model can be obtained in a similar fashion, maximizing (3) subject to the equilibrium constraint (14). Again, the relationship between the equilibrium parameters ψ, ζ, λ_i and the monopolist's choices m and σ_ϵ^2 are provided in the proof to Lemma 5. Table 2 presents a set of numerical results for different values of the risk-aversion parameter r, fixing the volatility of noise traders at $\sigma_z = 1$, for the market-orders model. As discussed in the paper, for r above the threshold $\hat{\kappa}^* \approx 3.1$, it is optimal to sell to a large number of agents ($m = 2000$ in Table 2). For values of the risk-aversion parameter below that cut-off, the monopolist can find it optimal to sell to anywhere between 1 and 5 agents.[3]

Noisy signals. We turn now to the case where the monopolist possess a noisy signal, i.e. the case $\sigma_\delta > 0$ where the signals are conditionally i.i.d. Lemmas 2 and 3 cover this case, but one can simplify the problem numerically significantly by exploiting the symmetry of the equilibrium. In particular, we have that $\beta_i = \beta$ and $\gamma_i = \gamma$ for all i, and furthermore $\beta_q = m\beta^2\sigma_\epsilon^2$ and $\beta_{iq} = \beta\sigma_\epsilon^2$. The price impact parameter λ_i in the limit-orders model, given by (33), can be written out explicitly as a function of γ, and v_i in (38) together with (5) allows us to write γ as a function of β and thereby have the system (4)-(5) collapse to a single equation

[2] We proxy numerically for N by using 2000 as the maximum number of agents the monopolist can sell her information to.

[3] Namely: for $\kappa < \kappa_1 \approx 0.19$ it is optimal to have $m = 1$; for $\kappa_1 < \kappa < \kappa_2 \approx 0.81$, it is optimal to have $m = 2$; for $\kappa_2 < \kappa < \kappa_3 \approx 1.56$, $m = 3$ maximizes profits; for $\kappa_3 < \kappa < \kappa_4 \approx 2.38$, the seller sets $m = 4$; and for $\kappa_4 < \kappa < \hat{\kappa}$, the seller sets $m = 5$.

for β. Numerically the problem therefore reduces, for each m, to a maximization problem over σ_ϵ^2 and β subject to a single non-linear constraint, just as in the previous case.

Table 3 contains the numerical results for the limit-orders model for the cases where $\sigma_\delta = 0.5$ and $\sigma_\delta = 1$. As in the case with perfect information, see Table 1, when $r = 0.1$ the monopolist finds it optimal to sell to a single agent without adding any noise. Profits are lower, as the noise in the seller's signal makes the trader value less the information received. In contrast to Table 1, we see that for moderate values of the risk-aversion parameter, namely when $r = 1$ in Table 3, the monopolist finds it optimal to sell to two agents. As r goes up eventually the seller optimally sells to as many agents as possible very noisy signals, i.e. for $r = 4$ as illustrated in Table 3.

Table 4 contains the numerical results for the market-orders case for the cases where $\sigma_\delta = 0.5$ and $\sigma_\delta = 1$. The results are qualitatively similar to those of Table 2, but now the monopolist may choose to sell to more agents. For example, when $r = 2$ and the monopolist has a perfect signal, she would sell to four agents, but with noisy signals it is optimal for her to sell to six agents. Once again, we find that for sufficiently high risk-aversion the monopolist finds it optimal to sell to many agents very noisy signals.

Asymmetric allocations. We finally turn to the fully unconstrained case, where the monopolist seller can choose from arbitrary positive semi-definite noise matrices $\mathbf{\Sigma}_\epsilon$. In order to keep the analysis tractable, we focus on the case where the seller offers two types of contracts, one to a set of m_A traders, to whom she offers signals of quality σ_A, and another to a set of m_B traders, to whom she offers signals of quality σ_B. We restrict the noise added to be conditionally i.i.d., and further assume that the monopolist has perfect information, $\sigma_\delta = 0$, but otherwise put no restrictions on the size of each group A and B.[4] Numerically, the system (4)-(5) to reduces to a set of four non-linear equations for the trading strategies of each group, (β_k, γ_k) for traders k in groups A or B. The problem in the limit-orders model therefore reduces to a maximization over σ_A and σ_B, as well as the four equilibrium parameters (β_A, γ_A) and (β_B, γ_B), subject to the four equilibrium constraints, (4)-(5) for each type of trader. Although non-trivial numerically, the problem is sufficiently well behaved so as to use standard techniques for its solution. In the market-orders model the problem reduces to a maximization over σ_A and σ_B, as well as the two equilibrium parameters β_A and β_B, subject to the two equilibrium constraints for groups A and B given by (7).

Table 5 reports the profits from selling to four different allocations: selling to one agent only ($m_A = 1$, $m_B = 0$), selling to two agents different quality information ($m_A = 1$ and $m_B = 1$), selling to one agent and to many other agents ($m_A = 1$ and $m_B = 2000$), and selling

[4]We have also analyzed the case where the monopolist sells three different types of signals, but we found two types of signals always dominates.

only to many agents ($m_A = 0$ and $m_B = 2000$). In these allocations the group with one single agent, A, receives a perfect signal, whereas the other group gets noisy signals (the table reports the optimal aggregate precision sold by the monopolist y_B). These four allocations are the only ones that we found to be optimal among a large class of possible allocations: we tried two hundred different allocations (m_A, m_B) randomly distributed over the set $\{1, 2, \ldots, 2000\}^2$, and optimized over (σ_A, σ_B) recording their profits. As the table shows, for intermediate values of the risk-aversion parameter it is optimal to have asymmetric allocations. For moderately low values of the risk-aversion parameter the monopolist sells perfect information to one single agent, giving a noisy signal to one other agent. For moderately high values of the risk-aversion parameter the monopolist optimally sells perfect information to one single agent, as well as giving noisy signals to a large number of agents. We also find that for low values of the risk-aversion parameter the monopolist optimally concentrates the information on one single trader, whereas for high values she finds it optimal to sell to as many agents as possible very noisy information, as in Table 1.

6 Proofs

Proof of Lemma 1.

For each trader, the certainty equivalent of wealth is the constant c_i that solves $\mathbb{E}\left[\mathcal{U}(\pi_i)\right] = \mathcal{U}(c_i)$, or

$$c_i = -\frac{1}{r_i} \log\left(-\mathbb{E}\left[\mathcal{U}(\pi_i)\right]\right). \tag{20}$$

We remind that trading profits are

$$\pi_i = \theta_i \left(X - P_x\right), \tag{21}$$

and from the first-order condition of an informed trader θ_i satisfies

$$\theta_i = \frac{\mathbb{E}[X - P_x|\mathcal{F}_i]}{r_i \mathrm{var}[X - P_x|\mathcal{F}_i] + \lambda_i}. \tag{22}$$

At the interim stage, profits are conditionally normal, so we can write

$$c_i = -\frac{1}{r_i} \log\left(\mathbb{E}\left[\mathbb{E}[\exp\left(-r_i\pi_i\right)|\mathcal{F}_i]\right]\right) = -\frac{1}{r_i} \log\left(\mathbb{E}\left[\exp\left(-r_i\chi_i\right)\right]\right), \tag{23}$$

where

$$\chi_i = \mathbb{E}[\pi_i|\mathcal{F}_i] - \frac{r_i}{2}\mathrm{var}(\pi_i|\mathcal{F}_i). \tag{24}$$

Defining $\eta_i \equiv \mathbb{E}[X - P_x|\mathcal{F}_i]$ and $\xi_i \equiv \text{var}[X - P_x|\mathcal{F}_i]$ we rewrite (24) using (21)-(22) as

$$\chi_i = \eta_i^2 \left(\frac{\lambda_i + r_i \xi_i/2}{(\lambda_i + r_i \xi_i)^2} \right). \tag{25}$$

In order to compute (23), notice that, as $P_x = \mathbb{E}[X|\mathcal{F}_u]$ and $\mathbb{E}[X] = 0$, we have that $\mathbb{E}[X - P_x] = 0$, implying $\mathbb{E}[\eta_i] = 0$. Moreover, by joint normality of (X, Y_i, P_x) and the fact that $\mathcal{F}_i = \sigma(Y_i, P_x)$ (limit-orders model) or $\mathcal{F}_i = \sigma(Y_i)$ (market-orders model), it follows that η_i is normally distributed. Standard results on the expectation of quadratic forms of Gaussian random variables imply that we can solve the expectation in (23) as[5]

$$\begin{aligned} \mathbb{E}\left[\exp\left(-r_i \chi_i\right)\right] &= \left(1 + 2 r_i \text{var}(\eta_i) \left(\frac{\lambda_i + r_i \xi_i/2}{(\lambda_i + r_i \xi_i)^2} \right)\right)^{-1/2} \\ &= \left(1 + 2r\mathbb{E}[\chi_i]\right)^{-1/2}, \end{aligned} \tag{27}$$

where the second equality follows by taking expectations in (25).

As a preliminary result for the next proofs of lemmas 4 and 5 we provide here a further characterization for (27): using the variance decomposition formula, $\text{var}[x] = \text{var}[\mathbb{E}[x|y]] + \mathbb{E}[\text{var}[x|y]]$, we can write

$$\begin{aligned} \text{var}(\eta_i) &= \text{var}[X - P_x] - \mathbb{E}\left[\text{var}[X - P_x|\mathcal{F}_i]\right] \\ &= \text{var}[X - P_x] - \text{var}[X - P_x|\mathcal{F}_i]. \end{aligned} \tag{28}$$

Moreover, as $\mathbb{E}[X - P_x|\mathcal{F}_u] = 0$, we have

$$\begin{aligned} \text{var}[X - P_x] &= \mathbb{E}\left[\text{var}[X - P_x|\mathcal{F}_u]\right] \\ &= \text{var}[X - P_x|\mathcal{F}_u]. \end{aligned} \tag{29}$$

Equations (28) and (29) imply

$$\text{var}(\eta_i) = \text{var}[X - P_x|\mathcal{F}_u] - \text{var}[X - P_x|\mathcal{F}_i],$$

[5] The expression follows from well known properties of multivariate normal distributions. In particular, if $X \sim \mathcal{N}(\mu, \Sigma)$ is an n-dimensional Gaussian random vector, $b \in \mathbb{R}^n$ is a vector, $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, it is well known that $\mathbb{E}\left[\exp\left(b^\top X + X^\top A X\right)\right]$ is well-defined if and only if $I - 2\Sigma A$ is positive definite, and further

$$\mathbb{E}\left[\exp\left(b^\top X + X^\top A X\right)\right] = |I - 2\Sigma A|^{-1/2} \exp\left[b^\top \mu + \mu^\top A \mu + \frac{1}{2}(b + 2A\mu)^\top (I - 2\Sigma A)^{-1} \Sigma (b + 2A\mu)\right]. \tag{26}$$

and therefore

$$\mathbb{E}\left[\exp\left(-r_i\chi_i\right)\right] = \left(1 + 2r_i\left(\text{var}[X-P|\mathcal{F}_u] - \text{var}[X-P|\mathcal{F}_i]\right)\left(\frac{\lambda_i + r_i\text{var}[X-P|\mathcal{F}_i]/2}{\left(\lambda_i + r_i\text{var}[X-P|\mathcal{F}_i]\right)^2}\right)\right)^{-1/2}. \tag{30}$$

□

Proof of Lemma 2.

We start with the usual conjecture that prices are a linear function of the order flow $\omega = \sum_{i=1}^N \theta_i + Z$, namely $P_x = \lambda\omega$ for some $\lambda \in \mathbb{R}$. We further conjecture that agent i's trading strategy is of the form $\theta_i = \beta_i Y_i - \gamma_i\omega$. We note that this implies

$$\omega = \frac{1}{(1+\gamma_s)}\left(\sum_{i=1}^N \beta_i Y_i + Z\right)$$

where $\gamma_s = \sum_{i=1}^N \gamma_i$.

The projection theorem yields the price impact parameter λ as a function of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$:

$$P_x = \mathbb{E}[X|\omega] = \lambda\omega = \frac{\beta_s(1+\gamma_s)}{(1+\sigma_\delta^2)\beta_s^2 + \beta_q + \sigma_z^2}\omega \tag{31}$$

where $\beta_s = \sum_{j=1}^N \beta_j$ and $\beta_q = \boldsymbol{\beta}^\top\boldsymbol{\Sigma}_\epsilon\boldsymbol{\beta}$, and $\beta_{qi} = \mathbf{1}_i^\top\boldsymbol{\Sigma}_\epsilon\boldsymbol{\beta}$ and $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^N$.

We note that order-flow can be written as

$$\omega = \theta_i + \sum_{j\neq i}\theta_j + Z = \frac{\theta_i + \sum_{j\neq i}\beta_j Y_j + Z}{\gamma_s - \gamma_i}, \tag{32}$$

so agent i's price impact λ_i is given

$$\lambda_i = \frac{\lambda}{(1+\gamma_s-\gamma_i)};. \tag{33}$$

The first-order condition for the optimal portfolio choice for agent i yields

$$\theta_i = \frac{\eta_i}{\lambda_i + r_i\xi_i}. \tag{34}$$

where $\eta_i \equiv \mathbb{E}[X - P_x|Y_i,\omega]$ and $\xi_i \equiv \text{var}(X - P_x|Y_i,\omega)$.

Using the projection theorem we can compute the conditional variance

$$\xi_i^{-1} = \text{var}(X|Y_i,\omega)^{-1} = 1 + \frac{\beta_q - 2\beta_s\beta_{qi} + \beta_s^2\sigma_i^2 + \sigma_z^2}{(\beta_q+\sigma_z^2)(\sigma_\delta^2+\sigma_i^2) + \beta_s^2\sigma_i^2 - \beta_{qi}^2 - 2\beta_s\beta_{qi}\sigma_\delta^2}, \tag{35}$$

where we remark ξ_i is given explicitly as a function of the primitives and $\boldsymbol{\beta}$.

Some tedious calculations further show that

$$\eta_i = \mathbb{E}[X - P_x | Y_i, \omega] = \alpha_i Y_i + \upsilon_i \omega \tag{36}$$

where

$$\alpha_i = \frac{\beta_q + \sigma_z^2 - \beta_s \beta_{qi}}{(\beta_s^2(1+\sigma_\delta^2) + \beta_q + \sigma_z^2)(1 + \sigma_\delta^2 + \sigma_i^2) - (\beta_s(1+\sigma_\delta^2) + \beta_{qi})^2} \tag{37}$$

$$\upsilon_i = \frac{(\beta_s \sigma_i^2 - \beta_{qi})(1+\gamma_s)}{(\beta_s^2(1+\sigma_\delta^2) + \beta_q + \sigma_z^2)(1 + \sigma_\delta^2 + \sigma_i^2) - (\beta_s(1+\sigma_\delta^2) + \beta_{qi})^2} - \lambda. \tag{38}$$

Using the conjecture on the trading strategy for each agent, as well as the first-order condition (34), one readily obtains the equilibrium conditions (4) and (5). □

Proof of Lemma 3.

We start with the usual conjecture that prices are a linear function of the order flow $\omega = \sum_{i=1}^N \theta_i + Z$, namely $P_x = \lambda\omega$ for some $\lambda \in \mathbb{R}$. We further conjecture that agent i's trading strategy is of the form $\theta_i = \beta_i Y_i$.

Using the projection theorem we have

$$P_x = \mathbb{E}[X|\omega] = \lambda\omega = \frac{\beta_s}{(1+\sigma_\delta^2)\beta_s^2 + \beta_q + \sigma_z^2}\omega \tag{39}$$

where $\beta_s = \sum_{j=1}^N \beta_j$ and $\beta_q = \boldsymbol{\beta}^\top \boldsymbol{\Sigma}_\epsilon \boldsymbol{\beta}$, where $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^N$.

The first-order condition for the optimal portfolio choice for agent i yields

$$\theta_i = \frac{\eta_i}{\lambda + r_i \xi_i}. \tag{40}$$

where $\eta_i \equiv \mathbb{E}[X - P_x|Y_i]$ and $\xi_i \equiv \text{var}(X - P_x|Y_i)$.

It is straightforward to verify that

$$\eta_i \equiv \mathbb{E}[X - P_x|Y_i] = \frac{(1 - \lambda((1+\sigma_\delta^2)\beta_s + \beta_{qi})}{1 + \sigma_\delta^2 + \sigma_i^2} Y_i. \tag{41}$$

with $\beta_{qi} = \mathbf{1}_i^\top \boldsymbol{\Sigma}_\epsilon \boldsymbol{\beta}$.

The standard projection theorem and (39) yield $\mathrm{var}(X - P_x|Y_i)$ as a function of $\boldsymbol{\beta}$:[6]

$$\xi_i \equiv \mathrm{var}(X - P_x|Y_i) = 1 - \frac{\beta_s^2}{\left(\sigma_\delta^2 + 1\right)\beta_s^2 + \sigma_z^2 + \beta_q} - \frac{\left(\sigma_z^2 + \beta_q - \beta_{qi}\beta_s\right)^2}{\left(\sigma_i^2 + \sigma_\delta^2 + 1\right)\left(\left(\sigma_\delta^2 + 1\right)\beta_s^2 + \sigma_z^2 + \beta_q\right)^2} \tag{42}$$

From the first-order condition (40) and (41) we have that the equilibrium trading strategy for the informed agent i is

$$\beta_i = \frac{1 - \lambda\left(\beta_{qi} + \beta_s\left(1 + \sigma_\delta^2\right)\right)}{\left(\lambda + r_i\xi_i\right)\left(1 + \sigma_\delta^2 + \sigma_i^2\right)} \tag{43}$$

Since we have expressed λ as a function of β_i in (39), and ξ_i has also been expressed as a function of $\boldsymbol{\beta}$'s in (42), the equilibrium is therefore fully characterized by the solution to the non-linear system (43). □

Proof of Lemma 4.

We solve for the equilibrium in Kyle (1989) setup with large number of uninformed speculators following Bernhardt and Taub (2006).[7] The procedure consists in solving an equivalent model in which informed speculators submit market orders conditioning on the price and assuming weak form efficiency:

$$P_x = \mathbb{E}[X|\omega] = \lambda\omega, \tag{44}$$

where ω is the order flow:

$$\omega = \sum_{i=1}^{m} \theta_i(Y_i, \omega) + Z,$$

and $\theta_i(Y_i, \omega) = \beta Y_i - \gamma\omega$ is speculator i's market order. Computing (44)

$$\lambda = \frac{\beta m s_\epsilon\left(1 + \gamma m\right)}{\beta^2 m(1 + m s_\epsilon) + \sigma_z^2 s_\epsilon}, \tag{45}$$

[6] The following conditional second moments involve some tedious calculations

$$\mathrm{var}(X|Y_i) = \frac{\sigma_i^2}{1 + \sigma_i^2};$$

$$\mathrm{var}(P_x|Y_i) = \frac{\lambda^2}{1 + \sigma_i^2}\left((\sigma_z^2 + \beta_q)(1 + \sigma_i^2) + \beta_s^2\sigma_i^2 - 2\beta_s\beta_{qi} - \beta_{qi}^2\right)$$

$$\mathrm{cov}(X, P_x|Y_i) = \frac{\lambda}{1 + \sigma_i^2}(\sigma_i^2\beta_s - \beta_{qi}).$$

[7] More precisely, we extend their results to the case of risk averse traders and to a different signal structure. The reader can verify that the system of equations in Lemma 4, that characterizes the endogenous variables ψ and ζ as a function of the exogenous parameters (κ, y, m), corresponds to the results in Kyle (1989).

and the variable ψ in (11) is given by

$$\psi = \frac{\beta^2 m}{\beta^2 m + \sigma_z^2 s_\epsilon}. \tag{46}$$

For speculator i, the order flow can be written as

$$\begin{aligned} \omega &= \theta_i(Y_i, \omega) + \sum_{j=1, j\neq i}^{m} \theta_j(Y_j, \omega) + Z \\ &= \frac{\theta_i(Y_i, \omega) + \beta \sum_{j=1, j\neq i}^{m} Y_j + Z}{1 + (m-1)\gamma}, \end{aligned}$$

and hence a speculator price impact is given by

$$\lambda_i \equiv \frac{dP_x}{d\theta_i} = \frac{\lambda}{1 + (m-1)\gamma}. \tag{47}$$

Speculator i chooses θ_i in order to maximize expected utility. The first order condition gives

$$\theta_i = \frac{\mathbb{E}[X|Y_i, \omega] - P_x}{r\tau_i^{-1} + \lambda_i}. \tag{48}$$

The projection theorem implies that the informed speculator posterior precision is given as in (11), with

$$\phi = \frac{(m-1)\beta^2}{(m-1)\beta^2 + \sigma_z^2 s_\epsilon}, \tag{49}$$

and

$$\mathbb{E}[X|Y_i, \omega] = \frac{\beta s_\epsilon Y_i + s_\epsilon \phi \left(\omega(1 + (m-1)\gamma) - \theta_i\right)}{\beta \tau_i}.$$

Substituting conditional moments into (48) and rearranging gives

$$\theta_i = \frac{1}{r\tau_i^{-1} + \lambda_i + \frac{\phi s_\epsilon \tau_i^{-1}}{\beta}} \left(\frac{s_\epsilon}{\tau_i} Y_i - \left(\lambda_i - \frac{s_\epsilon \phi}{\tau_i \beta} \right) \frac{\lambda}{\lambda_i} \omega \right).$$

Matching coefficients of the above expression with the conjectured strategy yields the following expressions for the undetermined coefficients β and γ:

$$\gamma = \frac{\lambda \tau_i \beta}{s_\epsilon} - \frac{\lambda \phi}{\lambda_i}, \tag{50}$$

$$\beta = \frac{s_\epsilon(1-\phi)}{r + \lambda_i \tau_i}. \tag{51}$$

Using (11), (45), (47) and (50) the price impact (47) can be written as

$$\lambda_i = \frac{m\beta s_\epsilon}{(m-1)\beta^2(1+ms_\epsilon)+\sigma_z^2 s_\epsilon}. \tag{52}$$

Since (45), (49) and (52) give λ, ϕ and λ_i as a function of β and primitive parameters, the equilibrium is thus characterized by a single non-linear equation for β, (51), from which all other endogenous quantities follow.

Following Kyle (1989), let us introduce a new parameter, denoted ζ, defined as the "informational incidence parameter"; ζ measures the (dollar) price variation that can be attributed to a dollar increase in the the informed speculator conditional expectation as a result of a larger realization of his signal. Notice that the speculator first-order condition can be written as

$$\theta_i = \frac{\mathbb{E}[X|Y_i,\omega] - \lambda_i\left(\beta\sum_{j\neq i}^{m} Y_j + Z\right)}{r\tau_i^{-1} + 2\lambda_i},$$

hence, by its definition we have

$$\zeta \equiv \frac{dP}{d\mathbb{E}[X|Y_i,\omega]} = \frac{dP}{d\theta_i}\frac{d\theta_i}{d\mathbb{E}[X|Y_i,\omega]} = \frac{\lambda_i\tau_i}{r+2\lambda_i\tau_i}. \tag{53}$$

We are now interested in defining the equilibrium as a function of the two endogenous parameters ζ and ψ. For this purpose, using (53) and (51) we can express

$$\frac{\zeta}{\tau_i} \equiv \frac{\lambda_i}{\tau_i\lambda_i + \frac{s_\epsilon(1-\phi)}{\beta}};$$

using the equilibrium conditions for λ_i and ϕ in (52) and (49),

$$\frac{\lambda_i}{\tau_i\lambda_i + \frac{s_\epsilon(1-\phi)}{\beta}} = \frac{m\beta^2}{m\beta^2(1+ms_\epsilon)+\sigma_z^2 s_\epsilon};$$

and from (46) and (11)

$$\frac{\psi}{\tau_u} = \frac{m\beta^2}{m\beta^2(1+ms_\epsilon)+\sigma_z^2 s_\epsilon}.$$

The last three equations imply the first equilibrium condition in (9)

$$\frac{\zeta}{\tau_i} = \frac{\psi}{\tau_u}.$$

Using definitions of τ_i and τ_u and (46) and (49) to eliminate ϕ from τ_i, the last equation can be expressed as

$$s_\epsilon = \frac{(m-\psi)(\zeta-\psi)}{\psi m(1+\psi(m-2)-\zeta(m-\psi))}. \tag{54}$$

To derive the second equilibrium condition in (9), using (53) we can rewrite (51) as

$$\beta = \frac{s_\epsilon}{r}\frac{(1-2\zeta)}{(1-\zeta)}(1-\phi); \tag{55}$$

then using (46) and (49) to eliminate β and ϕ from (55) yields the result. Therefore, (9) and (54) define the equilibrium in the endogenous ζ and ψ as a function of the exogenous (m, s_ϵ, κ).

Finally, some simple algebra shows that we can use (53), the definitions of τ_i and τ_u, and (30) to rewrite (1) as (13). □

Proof of Lemma 5.

We start by assuming weak form efficiency, namely that prices are of the form (44), where ω is the order flow:

$$\omega = \sum_{i=1}^{m} \theta_i(Y_i) + Z, \tag{56}$$

and $\theta_i(Y_i) = \beta Y_i$ is speculator i's market order. Given the conjectured strategy, the market maker's conditional expectation is based on

$$\omega = \beta \sum_{i=1}^{m} Y_i + Z = m\beta \left(X + \frac{1}{m} \left(\sum_{i=1}^{m} \epsilon_i + \frac{Z}{\beta} \right) \right).$$

Hence, computing the expected value of X given order flow yields

$$\lambda = \frac{\psi s_\epsilon}{\tau_u \beta}, \tag{57}$$

where

$$\tau_u \equiv \mathrm{var}[X|\omega]^{-1} = 1 + m s_\epsilon \psi,$$

and

$$\psi = \frac{\beta^2 m}{\beta^2 m + \sigma_z^2 s_\epsilon}. \tag{58}$$

Notice that for speculator i, the order flow can be written as

$$\omega = \theta_i(Y_i) + \sum_{j=1, j\neq i}^{m} \theta_j(Y_j) + u,$$

so that a speculator price impact is given by

$$\lambda_i \equiv \frac{dP_x}{d\theta_i} = \lambda. \tag{59}$$

Speculator FOC gives

$$\theta_i = \frac{\mathbb{E}[X - P_x|Y_i]}{r\tau_\pi^{-1} + \lambda}.$$

As speculator i only observes his private signal, the conditional moments of the payoff are given by

$$\begin{aligned} \mathbb{E}[X|Y_i] &= \frac{s_\epsilon Y_i}{\tau_i}, \\ \tau_i &= 1 + s_\epsilon; \end{aligned}$$

while the conditional moments of the profits are:

$$\begin{aligned} \mathbb{E}[X - P|Y_i] &= \mathbb{E}[X|Y_i](1 - \lambda(m-1)\beta) - \lambda\theta_i, \\ \tau_\pi &\equiv \text{var}[X - P|Y_i]^{-1} = \left(\tau_i^{-1}(1 - \lambda(m-1)\beta)^2 + \lambda\left(\frac{\beta^2(m-1)}{s_\epsilon} + \sigma_u^2\right)\right)^{-1}, \end{aligned}$$

using (57) and rearranging, the last formula simplifies into

$$\tau_\pi = \frac{\tau_i\tau_u^2}{\tau_i\tau_u - s_\epsilon(1-\psi)^2}. \tag{60}$$

Substituting for the conditional expectation of $X - P_x$ given Y_i into the speculator's first-order condition and rearranging gives

$$\theta_i = \frac{s_\epsilon(1 - \lambda(m-1)\beta)}{\tau_i\left(r\tau_\pi^{-1} + 2\lambda\right)} Y_i;$$

and solving for β gives

$$\beta = \frac{s_\epsilon(1 + s_\epsilon\psi)\tau_\pi}{\tau_i\tau_u(r + 2\lambda\tau_\pi)}, \tag{61}$$

where in the second equality we made use of (57). Since (58) and (57) give ψ and λ as a function of β, equation (61) characterizes the equilibrium value of β as the solution to a single non-linear equation.

Defining the parameter ζ as in the limit-orders case:

$$\zeta \equiv \frac{dP_x}{d\mathbb{E}[X|Y_i]} = \frac{dP_x}{d\theta_i}\frac{d\theta_i}{d\mathbb{E}[X|Y]} = \frac{\lambda\tau_\pi}{r + 2\lambda\tau_\pi}. \tag{62}$$

Using (62) we can express (61) as

$$\beta = \frac{s_\epsilon(1 + s_\epsilon\psi)\zeta}{\tau_i \tau_u \lambda}. \tag{63}$$

Using (57) and (59) to eliminate β in (63) and rearranging, we get the first equilibrium condition in (14)

$$\frac{\zeta}{\tau_i} = \frac{\psi}{1 + \psi s_\epsilon};$$

that can be solved for s_ϵ giving

$$s_\epsilon = \frac{\zeta - \psi}{\psi(1 - \zeta)}. \tag{64}$$

The second equilibrium condition in (14) can be obtained from (61) using (58) to eliminate β, and using (60) and (62) to eliminate λ_i and τ_π.

To derive the certainty equivalent expression in (18), as informed speculators condition only on signals, using (30) and (62), and the definitions of τ_π and τ_u into (2) yields the desired result. □

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Table 1: Optimal symmetric information sales with limit-orders

The table presents the optimal aggregate amount of noise, $y = ms_\epsilon$, for different values of the risk-aversion parameter r and the number of agents m, as well as the total profits for the monopolist, $\mathcal{C}$, in the limit-orders model when the monopolist has perfect information. The amount of noise trading is kept at $\sigma_z = 1$. The optimal information sales for each value of the risk-aversion parameter r are underlined.

	$r = 0.1$		$r = 0.5$		$r = 1$		$r = 2$	
m	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$
1	∞	<u>0.477</u>	∞	<u>0.405</u>	∞	<u>0.347</u>	∞	0.275
2	4.2	0.386	5.7	0.352	7.4	0.321	10.5	0.261
3	3.0	0.365	3.9	0.335	5.0	0.310	6.8	0.263
4	2.7	0.358	3.5	0.329	4.4	0.306	6.0	0.265
5	2.6	0.354	3.4	0.326	4.2	0.305	5.7	0.268
6	2.5	0.352	3.3	0.325	4.1	0.304	5.5	0.269
8	2.4	0.349	3.2	0.323	4.0	0.303	5.3	0.272
16	2.3	0.346	3.1	0.320	3.8	0.303	5.0	0.283
2000	2.3	0.343	3.0	0.318	3.7	0.302	4.8	<u>0.286</u>

	$r = 2.5$		$r = 3$		$r = 4$		$r = 8$	
m	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$
1	∞	0.251	∞	0.231	∞	0.201	∞	0.137
2	12.0	0.261	13.5	0.248	16.4	0.225	27.9	0.169
3	7.7	0.263	8.5	0.251	10.2	0.233	16.9	0.185
4	6.8	0.265	7.5	0.256	8.9	0.240	14.6	0.197
5	6.4	0.268	7.0	0.259	8.4	0.245	13.6	0.206
6	6.1	0.269	6.8	0.261	8.1	0.248	13.0	0.212
8	5.9	0.272	6.5	0.265	7.7	0.253	12.4	0.222
16	5.6	0.276	6.2	0.271	7.3	0.262	11.6	0.240
2000	5.4	<u>0.282</u>	5.9	<u>0.278</u>	6.9	<u>0.273</u>	11.0	<u>0.263</u>

Table 2: Optimal symmetric information sales with market-orders

The table presents the optimal aggregate amount of noise, $y = ms_\epsilon$, for different values of the risk-aversion parameter r and the number of agents m, as well as the total profits for the monopolist, $\mathcal{C}$, in the market-orders model when the monopolist has perfect information. The amount of noise trading is kept at $\sigma_z = 1$. In bold are the optimal information sales for each value of the risk-aversion parameter r.

	$r = 0.1$		$r = 0.5$		$r = 1$		$r = 2$	
m	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$
1	∞	**0.466**	∞	0.370	∞	0.299	∞	0.222
2	∞	0.454	∞	**0.397**	∞	0.346	∞	0.280
3	∞	0.422	∞	0.386	∞	**0.350**	∞	0.300
4	11.2	0.398	∞	0.368	∞	0.343	∞	**0.304**
5	6.2	0.385	12.3	0.356	26.8	0.333	∞	0.302
6	4.8	0.377	8.0	0.348	12.7	0.326	28.7	0.298
8	3.7	0.367	5.6	0.340	7.7	0.319	12.5	0.294
16	2.8	0.354	3.9	0.328	4.9	0.310	6.9	0.290
2000	2.3	0.343	3.0	0.318	3.7	0.302	4.8	0.286

	$r = 2.5$		$r = 3$		$r = 4$		$r = 8$	
m	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$
1	∞	0.199	∞	0.180	∞	0.153	∞	0.100
2	∞	0.257	∞	0.238	∞	0.210	∞	0.146
3	∞	0.281	∞	0.265	∞	0.238	∞	0.176
4	∞	0.289	∞	0.276	∞	0.254	∞	0.196
5	∞	**0.290**	∞	**0.279**	∞	0.261	∞	0.210
6	45.6	0.287	84.3	0.278	∞	0.263	∞	0.220
8	15.3	0.285	18.6	0.277	27.8	0.264	∞	0.230
16	7.9	0.283	8.9	0.277	10.9	0.268	21.7	0.244
2000	5.4	0.282	5.9	0.278	6.9	**0.273**	11.0	**0.263**

Table 3: Optimal sales with limit-orders and noisy signals

The table presents the optimal aggregate amount of noise, $y = ms_\epsilon$, for different values of the risk-aversion parameter r and the number of agents m, as well as the total profits for the monopolist, $\mathcal{C}$, in the limit-orders model. The amount of noise trading is kept at $\sigma_z = 1$. The optimal information sales for each value of the risk-aversion parameter r are underlined.

	$r = 0.1$		$r = 1$		$r = 2$		$r = 4$	
m	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$
A. Case $\sigma_\delta = 0.5$								
1	∞	0.385	∞	0.255	∞	0.185	∞	0.117
2	3.0	0.314	9.0	0.261	201.4	0.227	∞	0.178
3	2.0	0.296	4.3	0.249	7.9	0.224	40.2	0.193
4	1.9	0.290	3.6	0.246	5.9	0.223	14.0	0.197
6	1.7	0.285	3.2	0.243	4.9	0.224	9.1	0.202
8	1.7	0.282	3.1	0.242	4.5	0.224	7.8	0.206
16	1.6	0.279	2.9	0.240	4.1	0.226	6.6	0.211
2000	1.5	0.277	2.8	0.239	3.8	0.227	5.8	0.218
B. Case $\sigma_\delta = 1$								
1	∞	0.330	∞	0.203	∞	0.137	∞	0.080
2	2.3	0.271	19.5	0.224	∞	0.191	∞	0.135
3	1.6	0.255	4.2	0.214	12.8	0.193	∞	0.164
4	1.4	0.249	3.3	0.210	6.6	0.192	71.5	0.171
6	1.3	0.245	2.8	0.207	4.7	0.192	11.8	0.175
8	1.3	0.243	2.6	0.206	4.2	0.192	8.7	0.178
16	1.2	0.240	2.4	0.204	3.6	0.193	6.4	0.182
2000	1.2	0.237	2.3	0.203	3.3	0.193	5.2	0.186

Table 4: Optimal sales with market-orders and noisy signals

The table presents the optimal aggregate amount of noise, $y = ms_\epsilon$, for different values of the risk-aversion parameter r and the number of agents m, as well as the total profits for the monopolist, $\mathcal{C}$, in the market-orders model. The amount of noise trading is kept at $\sigma_z = 1$. The optimal information sales for each value of the risk-aversion parameter r are underlined.

	$r = 0.1$		$r = 1$		$r = 2$		$r = 4$	
m	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$	y	$\mathcal{C}$
A. Case $\sigma_\delta = 0.5$								
1	∞	$\underline{0.371}$	∞	0.217	∞	0.154	∞	0.100
2	∞	0.364	∞	0.259	∞	0.205	∞	0.149
3	∞	0.339	∞	$\underline{0.267}$	∞	0.225	∞	0.176
4	8.1	0.320	∞	0.265	∞	0.233	∞	0.192
6	3.3	0.303	12.2	0.255	349.0	$\underline{0.234}$	∞	0.207
8	2.6	0.295	6.6	0.250	15.1	0.231	∞	0.212
16	1.9	0.285	3.9	0.243	6.2	0.228	13.0	0.214
2000	1.5	0.275	2.8	0.237	4.0	0.225	6.2	$\underline{0.217}$
B. Case $\sigma_\delta = 1$								
1	∞	$\underline{0.316}$	∞	0.173	∞	0.119	∞	0.073
2	∞	0.311	∞	0.213	∞	0.165	∞	0.116
3	∞	0.291	∞	0.223	∞	0.186	∞	0.142
4	6.5	0.274	∞	$\underline{0.223}$	∞	0.195	∞	0.158
6	2.6	0.259	14.9	0.216	∞	$\underline{0.200}$	∞	0.174
8	2.0	0.253	6.3	0.212	24.1	0.198	∞	0.181
16	1.5	0.244	3.4	0.206	6.0	0.195	15.9	0.184
2000	1.2	0.236	2.3	0.201	3.4	0.192	5.6	$\underline{0.185}$

Table 5: Optimal sales with asymmetric allocations

The table presents the optimal aggregate precision sold to group B, y_B, as well as the consumer surplus, in four types of allocations: selling to one agent only ($m_A = 1$, $m_B = 0$), selling to two agents different quality information ($m_A = 1$ and $m_B = 1$), selling to one agent and to many other agents ($m_A = 1$ and $m_B = 2000$), and selling only to many agents ($m_A = 0$ and $m_B = 2000$). In these allocations the group with one single agent, A, receives a perfect signal, whereas the other group gets noisy signals (the aggregate precision sold by the monopolist is given by y_B).

	(m_A, m_B)	$(1,0)$	$(1,1)$	$(1,2000)$	$(0,2000)$
$r = 0.1$	y_B	–	0.00	0.00	2.25
	$\mathcal{C}$	<u>0.477</u>	0.477	0.477	0.343
$r = 1$	y_B	–	0.42	0.36	3.65
	$\mathcal{C}$	0.347	<u>0.349</u>	0.349	0.302
$r = 1.5$	y_B	–	0.94	0.82	4.25
	$\mathcal{C}$	0.305	<u>0.314</u>	0.313	0.293
$r = 1.75$	y_B	–	1.19	1.04	4.53
	$\mathcal{C}$	0.289	0.300	<u>0.300</u>	0.289
$r = 2$	y_B	–	1.44	1.26	4.81
	$\mathcal{C}$	0.275	0.287	<u>0.288</u>	0.286
$r = 4$	y_B	–	3.32	2.94	4.53
	$\mathcal{C}$	0.201	0.226	0.231	<u>0.273</u>