Monotonicity in direct revelation mechanisms*

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Abstract

This paper studies a standard screening problem where the principal's allocation rule is multi-dimensional, and the agent's private information is a one-dimensional continuous variable. Under standard assumptions, that guarantee monotonicity of the allocation rule in one-dimensional mechanisms, it is shown that the optimal allocation will be non-monotonic in a (weakly) generic sense once the principal can use all screening variables. The paper further gives conditions on the model's parameters that guarantee that non-monotonic allocation rules will be optimal.

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1 Introduction

This paper studies a screening problem when the asymmetric information faced by the principal can be described by a one-dimensional variable, but allocation rules may be multi-dimensional. It is well known, since at least Matthews and Moore (1987), that in a multi-dimensional context it is possible for some element of the allocation vector to be non-monotonic under the optimal schedule. In particular, the authors presented a three-type example, in the context of a monopolist screening through warranties and quality, in which the optimal allocation was a non-monotone function of the private information of the agent, and offered a set of necessary conditions for non-monotonic optimal contracts.[1] The purpose of this paper is to extend this weak genericity result[2] to the standard setup with quasi-linear preferences, and find a set of simple conditions under which the optimal mechanism will exhibit the type of non-monotonicity properties first discussed by Matthews and Moore (1987). It is shown that either (1) a total surplus function with negative cross-partial derivatives, or (2) a marginal utility (with respect to information) for the agent with positive cross-partial derivatives, can generate optimal non-monotone allocation rules.

2 The model

Consider a relationship between two parties, who negotiate over the exchange of an allocation, characterized by the vector $x \in \mathbb{R}^n$, and a payment $t \in \mathbb{R}$. One of the parties, the agent, has private information regarding one of the variables that affect the gains from trade. This private information is assumed to be measured by the parameter $\theta \in [\underline{\theta}, \bar{\theta}]$. The principal's beliefs about θ are described by the density function $g(\theta)$, which has associated distribution function $G(\theta)$. The inverse of the hazard rate will be denoted by $\mu(\theta) \equiv (1 - G(\theta))/g(\theta)$, and assumed to be bounded.

Both the principal and the agent have quasi-linear preferences. The agent's utility function is denoted by $u(x, \theta) + t$, and the principal's utility function is given by $v(x, \theta) - t$. It will be assumed that $u_\theta \geq 0$, so θ orders the willingness to trade by the agent. By the revelation principle we can restrict attention to direct mechanisms $\{x(\theta), t(\theta)\}$ that induce truthful revelation by the agent. It is assumed that the principal has all the bargaining power in the negotiations, i.e. that she makes a take-it-or-leave-it offer to the agent. The agent's reservation utility is

[1] See Theorem 3 and Example 1 in Matthews and Moore (1987). Another example of this non-monotonicity result, in a similar spirit to that in Matthews and Moore (1987), is provided in van Egteren (1996).

[2] A property is said to hold in a weakly generic sense if it holds for an open set of the parameter values of the model.

denoted by $\bar{u}$.

The problem that the principal faces is

$$\max_{\{x(\theta),t(\theta)\}} \int_{\underline{\theta}}^{\bar{\theta}} (v(x(\theta),\theta) - t(\theta))\, g(\theta) d\theta, \tag{1}$$

such that

$$u(x(\theta),\theta) + t(\theta) \geq \bar{u}, \qquad \forall \theta \in [\underline{\theta}, \bar{\theta}]; \tag{2}$$

$$u(x(\theta),\theta) + t(\theta) \geq u(x(\hat{\theta}),\theta) + t(\hat{\theta}), \qquad \forall \theta, \hat{\theta} \in [\underline{\theta}, \bar{\theta}]. \tag{3}$$

The principal maximizes her expected utility subject to the participation constraint for the agent, equation (2), and the incentive compatibility constraint (3), which requires that the agent reveals his type truthfully in the direct revelation mechanism.

Define the total surplus from trade as $S(x,\theta) \equiv v(x,\theta) + u(x,\theta)$. The following assumptions, which will be referred to as "standard assumptions," will be made throughout the paper:[3] (1) single-crossing property: $u_{\theta x_i} \geq 0$, for $i = 1, \ldots, n$; (2) monotone-hazard rate condition, $\mu'(\theta) < 0$; (3) monotonicity conditions: $S_{x_i\theta} \geq 0$, and $u_{\theta\theta x_i} \leq 0$, for $i = 1, \ldots, n$; (4) differentiability: $x(\theta)$ and $t(\theta)$ are continuous and differentiable. The single-crossing property and the monotone-hazard rate condition are standard in the literature. As discussed below the "monotonicity conditions" are not necessary for the main results: they simply guarantee that, when the principal has only one variable x_i for screening, the optimal mechanism satisfies $dx_i/d\theta > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, i.e. the monotonicity constraint does not bind.[4] The differentiability conditions are made for simplicity, since the focus of the paper is on monotonicity properties of the optimal solutions, and such properties are easier to express in terms of derivatives.

3 Implementable mechanisms

The following proposition gives a set of necessary and sufficient conditions for implementability of a given mechanism. The proof follows closely similar characterizations in the literature.[5]

Proposition 1. *The following two conditions are sufficient for a mechanism $\{x(\theta), t(\theta)\}$ to*

[3] The standard notation $f_x(\cdot)$ (or $f'(x)$ if $x \in \mathbb{R}^1$) will be used for the derivative of a function $f(\cdot)$ with respect to x. For notational ease the arguments of the functions will be omitted where there is no room for ambiguity.

[4] If these conditions are not met then bunching may occur, i.e. the optimal allocations may have $dx_i/d\theta = 0$ for a subset of the types of the agent (see the discussions in Mussa and Rosen (1978), Guesnerie and Laffont (1984) or Maskin and Riley (1984)).

[5] All proofs are relegated to the Appendix.

be implementable:

$$u_x(x(\theta),\theta)^\top \frac{dx}{d\theta}(\theta) + \frac{dt}{d\theta}(\theta) = 0; \qquad \forall \theta \in \Theta; \tag{4}$$

$$u_{x\theta}(x(\hat\theta),\theta)^\top \frac{dx}{d\theta}(\hat\theta) \geq 0; \qquad \forall \hat\theta, \theta \in \Theta. \tag{5}$$

A necessary condition for implementability is that (4) holds, and

$$u_{x\theta}(x(\theta),\theta)^\top \frac{dx}{d\theta}(\theta) \geq 0; \qquad \forall \theta \in \Theta. \tag{6}$$

If $u(x,\theta)$ is linear in θ, conditions (4) and (6) are necessary and sufficient for implementability.

The necessary conditions are well known.[6] The above proposition is not as sharp as the standard characterization of implementable mechanisms, since (5) is a two-dimensional constraint: in order to get sufficiency it is necessary to check a two-dimensional constraint. The literature only points out to $dx_i/d\theta \geq 0$ as a sufficient condition for implementability: together with the assumption of the single-crossing property for each x_i this makes (5) hold for all $\hat\theta, \theta \in \Theta$. Note that even though this monotonicity restriction on each x_i yields sufficiency by analyzing n one-dimensional constraints, it also unnecessarily rules out a large class of implementable mechanisms. The proposition further gives a condition under which we can reduce (5) to a one-dimensional constraint, linearity of the utility function $u(x,\theta)$ in θ.[7]

Incentive compatibility does not impose monotonicity on each element of the allocation rule x, but it does force the mechanism to satisfy a *weighted monotonicity constraint*. Dividing (6) by $\sum_{i=1}^n u_{x_j\theta}(x(\theta,\theta))$ we have the following necessary and sufficient condition for implementability in the linear case

$$\sum_{i=1}^n w_i(\theta) x_i'(\theta) \geq 0; \tag{7}$$

where $w_i(\theta) \equiv u_{x_i\theta}/\sum_j u_{x_j\theta} > 0$ are the weights on each element on the allocation rule x. Few general statements can be made about the implications of the incentive compatibility with regards to the monotonicity of elements of the allocation vector. Incentive compatibility only implies that at least one of the elements of the allocation rule will be non-decreasing.[8] The usual implication of monotonicity of the allocation vector from incentive compatibility only

[6] See Caillaud, Guesnerie, Rey, and Tirole (1988), Fudenberg and Tirole (1991), Guesnerie and Laffont (1984), or Mirman and Sibley (1980).

[7] This linearity assumption also appears in the discussion of the "taxation principle" in Rochet (1987).

[8] In particular, for all θ, there exists i such that $x_i'(\theta) \geq 0$. Moreover, if for some i the mechanism is such that $x_i'(\theta) < 0$, then there exists j such that $x_j'(\theta) > 0$.

follows in the one-dimensional case: in general conclusions about monotonicity of the allocation rule cannot be drawn without solving for the optimal mechanism.

4 Optimal mechanisms

The next proposition reduces the principal's problem to that of maximizing a distorted surplus function.

Proposition 2. *The principal's problem reduces to*

$$\max_{x} \int_{\underline{\theta}}^{\bar{\theta}} \Phi(x, s) ds$$

where

$$\Phi(x, \theta) \equiv S(x, \theta) - \mu(\theta) u_\theta(x, \theta); \tag{8}$$

such that the constraint (5) holds.

If the constraint (5) does not bind, the optimal allocation x satisfies

$$\frac{dx}{d\theta} = - \left[S_{xx} - \mu u_{\theta xx} \right]^{-1} \left(S_{x\theta} - \mu' u_{\theta x} - \mu u_{\theta\theta x} \right). \tag{9}$$

The optimal allocation rule x has an element with $dx_i/d\theta < 0$ in a weakly generic sense.

The result in the above proposition has a similar flavor to those in the literature: the principal maximizes the "virtual surplus" function $\Phi(\cdot)$, with $\mu(\theta)u_\theta(x, \theta)$ measuring informational costs. The main qualitative difference stems from the constraint (5), which in the one-dimensional case implied monotonicity of the allocation rule, whereas this is not the case in the multi-dimensional problem.

Assuming an interior solution, i.e. that the "monotonicity constraint" (5) is not binding, the proposition also gives the sign of the derivative of the optimal allocation rule with respect to the private information parameter θ. From this proposition it is straightforward to get conditions for the optimal allocation x to be non-monotone in θ. Consider, for example, the two-dimensional case. Condition (9) implies that, when $u(x, \theta)$ is linear in θ and (6) does not bind,[9] $dx_1/d\theta < 0$ if and only if

$$-(S_{x_2x_1} - \mu u_{\theta x_1 x_2}) > -(S_{x_2x_2} - \mu u_{\theta x_2 x_2})\frac{(S_{x_1\theta} - \mu' u_{x_1\theta} - \mu u_{\theta\theta x_1})}{(S_{x_2\theta} - \mu' u_{\theta x_2} - \mu u_{\theta\theta x_2})}. \tag{10}$$

The right-hand side in equation (10) is positive under the "standard assumptions," so in order to have a decreasing x_1 it is necessary to have $\mu u_{\theta x_1 x_2} - S_{x_1x_2} > 0$. It is worth noticing that in the case where $S_{x_1x_2} = 0$, which generates a monotonically increasing allocation in a first-best world, the optimal allocation will be non-monotonic as long as $u_{\theta x_1 x_2} > 0$ and either this quantity or $\mu(\theta)$ are large. In order to gain some intuition, note that the only motivation for having a non-increasing allocation rule comes from informational rents considerations when $S_{x_1x_2} = 0$. A change in x_1 has a direct informational cost, as in the one-dimensional model: if we decrease x_1 informational rents are cut. On the other hand, there is an indirect effect on x_2 due to a change in x_1, which also affects informational rents. It is straightforward to check that dx_2/dx_1 has the opposite sign of $u_{\theta x_1 x_2}$, so when $u_{\theta x_1 x_2}$ is positive there is an indirect informational cost associated with cutting x_1. This in itself may be sufficient to generate a decreasing allocation rule x_1.

It is also worth noticing that if $S_{x_1x_2} < 0$, it is possible for the optimal allocation to satisfy $dx_i/d\theta < 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. The intuition for the result is simple: if there are gains for the principal from having an allocation that is decreasing, then this could be the outcome in this multi-dimensional setting, since incentive compatibility does not rule out non-increasing allocation rules. Note that this result holds while the single-crossing property is satisfied for all elements of the allocation vector, which guarantees that the solution in the one-dimensional case will be strictly increasing.

[9] It is important to note that the restriction to parameter values for which (5) does not bind does not hamper the non-monotonicity results being discussed. When this constraint binds it is immediate from (5) that either: (1) an element of the allocation vector is decreasing in θ, or (2) there is trivial screening ($x_i'(\theta) = 0$ for all i).

Appendix

Proof of Proposition 1. The agent's optimization problem is clearly characterized by the first-order condition (4). The second-order necessary condition is

$$\frac{dx}{d\theta}^{\top} u_{xx} \frac{dx}{d\theta} + u_x \frac{d^2x}{d\theta^2} + \frac{d^2t}{d\theta} \leq 0; \quad \forall \theta \in \Theta. \tag{11}$$

Since the first-order condition for the agent's optimization problem holds as an identity for all θ, we can total differentiate the expression with respect to θ, and using (11) we obtain the necessary condition (6).

To see that the mechanism is globally incentive compatible under (4) and (5), suppose not. Then there exists $\hat{\theta}$ such that $U(\hat{\theta}, \theta) > U(\theta, \theta)$, where $U(\hat{\theta}, \theta) \equiv u(x(\hat{\theta}), \theta) + t(\hat{\theta})$, so that $\int_{\theta}^{\hat{\theta}} \frac{\partial U}{\partial \hat{\theta}}(y, \theta) > 0$. If $\hat{\theta} > \theta$, then from (5) we have that

$$\frac{\partial U}{\partial \hat{\theta}}(\hat{\theta}, \theta) = u_x(x(\hat{\theta}), \theta)^{\top} \frac{dx}{d\theta}(\hat{\theta}) + \frac{dt}{d\theta}(\hat{\theta}) \leq u_x(x(\hat{\theta}), \hat{\theta})^{\top} \frac{dx}{d\theta}(\hat{\theta}) + \frac{dt}{d\theta}(\hat{\theta}) = \frac{\partial U}{\partial \hat{\theta}}(\hat{\theta}, \hat{\theta});$$

which implies that $\int_{\theta}^{\hat{\theta}} \frac{\partial U}{\partial \hat{\theta}}(y, y)dy > 0$; which contradicts (4). An identical argument follows when $\hat{\theta} < \theta$, which shows that indeed (4) and (5) are sufficient for global incentive compatibility.

In order to see that (4) and (6) are necessary and sufficient in the linear case, note that the linearity assumption makes $u_{x\theta}(x(\hat{\theta}), \theta)$ independent of θ, so (5) reduces to (6). □

Proof of Proposition 2. Expression (8) follows by solving for the transfers as a function of the allocation from (4) and then integrating by parts the principal's objective function. The first-order condition to the principal's optimization problem is $S_x(x, \theta) - \mu(\theta) u_{x\theta}(x, \theta) = 0$. Applying the implicit function theorem to the above expression yields (9). In order to proof weak genericity, let us consider the two-dimensional case discussed in the text, and further assume $u(x, \theta)$ is linear in θ. All we need to show is that there is an open set of parameter values such that (10) holds, while the second-order conditions and the weighted monotonicity constraint (6) are not violated. Note that the second-order conditions to the principal's problem are $S_{x_1x_1} - \mu u_{\theta x_1x_1} \leq 0$; $S_{x_2x_2} - \mu u_{\theta x_2x_2} \leq 0$; $(S_{x_1x_1} - \mu u_{\theta x_1x_1})(S_{x_2x_2} - \mu u_{\theta x_2x_2}) - (S_{x_1x_2} - \mu u_{\theta x_1x_2})^2 \geq 0$. Take $|S_{x_1x_2} - \mu u_{\theta x_1x_2}|$ large and at the same time raise $S_{x_1x_1}$ and $u_{\theta x_2}$, in order to assure that (6) does not bind and that the second order conditions are satisfied. One can easily verify under such conditions $dx_1/d\theta$, as given in (10), is indeed negative. □

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