4. Options Markets

4.5. Black-Scholes
Stock Returns $\frac{\Delta S}{S}$

- are normally distributed
- with a mean of $\mu \Delta t$
- and a standard deviation of $\sigma \sqrt{\Delta t}$
Intuition

• Just like binomial trees: form a riskless portfolio

• Except that the step size $\Delta t$ is very small
Black-Scholes Formulas

\[ c = SN(d_1) - Xe^{-rT}N(d_2) \]

\[ p = Xe^{-rT}N(-d_2) - SN(-d_1) \]

\[
d_1 = \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}
\]

\[
d_2 = \frac{\ln \frac{S}{X} + (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}
\]

\[
= d_1 - \sigma \sqrt{T}
\]
The Cumulative Distribution Function $N(d)$

$N(d)$ is the probability that a variable

- that is normally distributed
- with a mean of 0
- and a standard deviation of 1

is less than $d$

A symmetric distribution implies that $N(d) + N(-d) = 1$
Black-Scholes and Lower Bounds

• As $S$ becomes very large,
    $d_1$ and $d_2$ become very large
    $N(d_1)$ and $N(d_2)$ tend to 1
    $N(-d_1)$ and $N(-d_2)$ tend to 0
    $c$ tends to $S - X e^{-rT}$
    $p$ tends to 0

• As $S$ becomes very small,
    $d_1$ and $d_2$ become very small
    $N(d_1)$ and $N(d_2)$ tend to 0
    $N(-d_1)$ and $N(-d_2)$ tend to 1
    $c$ tends to 0
    $p$ tends to $X e^{-rT} - S$
Implied Volatility

- An option price $c$ or $p$ can be solved using the Black-Scholes formula with:
  - the stock price $S$,
  - the strike price $X$,
  - the risk-free rate $r$,
  - the time to maturity $T$, and
  - the volatility $\sigma$.

- Alternatively, the formula can be used to find the volatility implied by:
  - the option price $c$ or $p$,
  - the stock price $S$,
  - the strike price $X$,
  - the risk-free rate $r$,
  - and the time to maturity $T$. 
Black-Scholes Formulas for a Dividend-Paying Stock

\[ c = (S - D)N(d_1) - Xe^{-rT}N(d_2) \]
\[ p = Xe^{-rT}N(-d_2) - (S - D)N(-d_1) \]

\[ d_1 = \frac{\ln \left( \frac{S-D}{X} \right) + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \]
\[ d_2 = \frac{\ln \left( \frac{S-D}{X} \right) + (r - \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \]
\[ = d_1 - \sigma \sqrt{T} \]
Black’s Approach

• The Black-Scholes formulas only price European options, and American calls on non-dividend-paying stocks.

• Black’s approach to approximate the American call price $C$ on a dividend-paying stock:
  \[
  \max\{c, c \text{ with } T = \text{final ex-div date}\}
  \]
Homework

1. (Baby Hull 12.13, Papa Hull 13.13) What is the price of a European call option on a non-dividend-paying stock when the stock price is $52, the strike price is $50, the risk-free interest rate is 12 percent per annum, the volatility is 30 percent per annum, and the time to maturity is three months?

2. (Baby Hull 12.14, Papa Hull 13.14) What is the price of a European put option on a non-dividend-paying stock when the stock price is $69, the strike price is $70, the risk-free interest rate is 5 percent per annum, the volatility is 35 percent per annum, and the time to maturity is six months?

3. (Baby Hull 12.25, Papa Hull 13.29) Consider an option on a non-dividend-paying stock when the stock price is $30, the exercise price is $29, the risk-free interest rate is 5 percent per annum, the volatility is 25 percent per annum, and the time to maturity is four months.
   (a) What is the price of the option if it is a European call?
   (b) What is the price of the option if it is an American call?
   (c) What is the price of the option if it is a European put?
   (d) Verify that put-call parity holds.

4. (Baby Hull 12.26, Papa Hull 13.30) Assume that the stock in the previous problem is due to go ex-dividend in 1.5 months. The expected dividend is 50 cents.
   (a) What is the price of the option if it is a European call?
   (b) What is the price of the option if it is a European put?