4. Options Markets

4.4. Binomial Trees
What is the payoff of a call option with a strike price of 42?
Riskless Portfolio

• Portfolio:
  buy \( \Delta \) shares & write a call

\[
+44\Delta - 1 \times 2 \\
-40\Delta + c \\
+36\Delta - 1 \times 0
\]

• Riskless if
  \[+44\Delta - 1 \times 2 = +36\Delta - 1 \times 0\]
What is $c$?

$$r = 0.12$$

$$\text{step} = \frac{3}{12}$$
Binomial Means Two, Take 2

$S$ → $Su$

$S$ → $Sd$
Riskless Portfolio, Take 2

• Portfolio:
  buy $\Delta$ shares & write an option

\[
\begin{align*}
  +S u \Delta - 1 \times f_u \\
  -S \Delta + f \\
  +S d \Delta - 1 \times f_d
\end{align*}
\]

• Riskless if
  \[+S u \Delta - 1 \times f_u = +S d \Delta - 1 \times f_d\]

\[\Delta = \frac{f_u - f_d}{S u - S d}\]
What is $f$? Take 2

\[ 0 = -S\Delta + f + (Su\Delta - f_u)e^{-rT} \]
\[ = -S\Delta + f + (Sd\Delta - f_d)e^{-rT} \]

or

\[ f = S\Delta - (Su\Delta - f_u)e^{-rT} \]
\[ = S\Delta - (Sd\Delta - f_d)e^{-rT} \]
Risk-Neutral Probabilities

\[ f = S \Delta - (S u \Delta - f_u) e^{-rT} \]

\[ = S \frac{f_u - f_d}{u - d} - (S u \frac{f_u - f_d}{S u - S d} - f_u) e^{-rT} \]

\[ = \frac{f_u - f_d}{u - d} - \left( \frac{u (f_u - f_d)}{u - d} - f_u \right) e^{-rT} \]

\[ = \left[ \frac{e^{rT} (f_u - f_d)}{u - d} - \left( \frac{u (f_u - f_d)}{u - d} - \frac{f_u (u - d)}{u - d} \right) \right] e^{-rT} \]

\[ = \left[ \left( \frac{e^{rT} - d}{u - d} \right) f_u + \left( \frac{u - e^{rT}}{u - d} \right) f_d \right] e^{-rT} \]

\[ = \left[ \left( \frac{e^{rT} - d}{u - d} \right) f_u + \left( 1 - \frac{e^{rT} - d}{u - d} \right) f_d \right] e^{-rT} \]

\[ = [p f_u + (1 - p) f_d] e^{-rT} \]

where \( p \equiv \frac{e^{rT} - d}{u - d} \) can be interpreted as the risk-neutral probability of an up movement.
Now Two Steps

\[ X = 42 \]
\[ r = 0.12 \]
\[ \text{step} = \frac{3}{12} \]
what is \( c \)?
What is $p$?

$X = 26$
$r = 0.05$
step = 1
What is $P$?

$X = 26$

$r = 0.05$

step = 1
Homework

1. (Hull 11.9) A stock price is currently $50. It is known that at the end of two months it will be either $53 or $48. The risk-free interest rate is 10 percent per annum with continuous compounding. What is the value of a two-month European call option with a strike price of $49?

2. (Hull 11.10) A stock price is currently $80. It is known that at the end of four months it will be either $75 or $85. The risk-free interest rate is 5 percent per annum with continuous compounding. What is the value of a four-month European put option with a strike price of $80?

3. (Hull 11.17) A stock price is currently $40. Over each of the next two three-month periods it is expected to go up by 10 percent or down by 10 percent. The risk-free interest rate is 12 percent per annum with continuous compounding.
   (a) What is the value of a six-month European put option with a strike price of $42?
   (b) What is the value of a six-month American put option with a strike price of $42?