2. Futures and Forward Markets

2.2. Pricing
An Arbitrage Opportunity?

- Gold spot price: $300 per oz
- Gold 1-year forward price: $325 per oz
- Time-to-delivery: one year
- Rate of interest per annum (with annual compounding): 5%
No Arbitrage

• Spot price: $S$
• Forward price: $F$
• Time-to-delivery: $T$
• Rate of interest per annum (with annual compounding): $r$

$$\rightarrow F = S(1 + r)^T$$
Arbitrage Strategy

• buy low: buy gold spot
• sell high: short gold forward

$$\text{Arbitrage}_T = -300(1.05) + 325 = 10$$

$$\text{Arbitrage}_0 = \frac{10}{1.05} = 9.52$$
Profit$_T$ from a Short Forward Position

At the delivery time $T$ where $K$ denotes the delivery price,

- buy spot: $-S_T$
- short forward: $+K = 315$
- profit$_T = -S_T + K$

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Profit$_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>305</td>
<td>$-305 + 315 = 10$</td>
</tr>
<tr>
<td>310</td>
<td>$-310 + 315 = 5$</td>
</tr>
<tr>
<td>315</td>
<td>$-315 + 315 = 0$</td>
</tr>
<tr>
<td>320</td>
<td>$-320 + 315 = -5$</td>
</tr>
<tr>
<td>325</td>
<td>$-325 + 315 = -10$</td>
</tr>
</tbody>
</table>
Profit$_T$ from a Long Forward Position

At the delivery time $T$ where $K$ denotes the delivery price,

- sell spot: $+S_T$
- long forward: $-K = 315$
- profit$_T = S_T - K$

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Profit$_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>305</td>
<td>$305 - 315 = -10$</td>
</tr>
<tr>
<td>310</td>
<td>$310 - 315 = -5$</td>
</tr>
<tr>
<td>315</td>
<td>$315 - 315 = 0$</td>
</tr>
<tr>
<td>320</td>
<td>$320 - 315 = 5$</td>
</tr>
<tr>
<td>325</td>
<td>$325 - 315 = 10$</td>
</tr>
</tbody>
</table>
Value of a Forward Contract $f$

The value of a long position is:

- $f_T = S_T - K$ at the delivery time
- $f_t = \frac{F_t - K}{1 + r_{t,T}}$ before the delivery
- $f_0 = \frac{K - K}{1 + r_{0,T}} = 0$ at the initial time

The value of a short position is:

- $f_T = -S_T + K$ at the delivery time
- $f_t = \frac{-F_t + K}{1 + r_{t,T}}$ before the delivery
- $f_0 = \frac{-K + K}{1 + r_{0,T}} = 0$ at the initial time
Spot, Forward, and Delivery Prices

- \( F_0 = K \rightarrow f_0 = 0 \)
- \( F_T = S_T \)
- \( F < S \): backwardation
  - \( F > S \): contango
  - \( b = S - F \): basis
- \( \frac{\partial F}{\partial T} > 0 \): normal market
- \( \frac{\partial F}{\partial T} < 0 \): inverted market
Compounding Frequency

How much do you have at the end of one year if you invest $100,000 now for one year at 5% per annum with

- annual compounding?
  \[ 100,000(1 + 0.05) = 105,000 \]

- semi-annual compounding?
  \[ [100,000(1 + 0.025)](1 + 0.025) = 105,062.50 \]

- continuous compounding?
  \[ 100,000e^{0.05} = 105,127.11 \]
Conversion between Compounding Frequencies

Let $r_c$ denote the continuously compounded rate per annum and $r_m$ the equivalent rate with compounding $m$ times per year

\[ e^{r_c T} = \left( 1 + \frac{r_m}{m} \right)^{mT} \]

\[ \leftrightarrow r_c = m \ln \left( 1 + \frac{r_m}{m} \right) \]

\[ \leftrightarrow r_m = m \left( e^{\frac{r_c}{m}} - 1 \right) \]
Conversion Examples

1. The interest rate is 10% per annum with semi-annual compounding. What is the equivalent annual rate with continuous compounding?

2. The interest rate is 8% per annum with continuous compounding. What is the equivalent annual rate with annual compounding?
Compounding and Discounting

- How much do you have at the end of one year if you invest $100,000 now for one year at 5% per annum with continuous compounding?
  \[ 100,000e^{0.05} = 105,127.11 \]

- How much do you need to invest now for one year at 5% per annum with continuous compounding if you want to have $100,000 at the end of the year?
  \[ 100,000e^{-0.05} = 95,122.94 \]
Forward Pricing

- Spot price: $S$
- Forward price: $F$
- Time-to-delivery: $T$
- Rate of interest per annum: $r$
  1. with annual compounding
     \[ F = S(1 + r)^T \]
  2. with continuous compounding
     \[ F = Se^{rT} \]
**Investment Asset that Provides No Income**

<table>
<thead>
<tr>
<th>Want asset at $T$</th>
<th>Cost at 0</th>
<th>Cost at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy forward</td>
<td>0</td>
<td>$-F$</td>
</tr>
<tr>
<td>buy spot</td>
<td>$-S$</td>
<td>0</td>
</tr>
</tbody>
</table>

$$F = Se^{rT}$$
Example of an Investment Asset that Provides No Income

What is the price of a 2-month forward contract to purchase a pure discount bond that will mature six months from today, given that the current price of the bond is $950 and the 2-month risk-free (continuously compounded) interest rate is 5% per annum?
Investment Asset that Provides a Known Income \( I \)

<table>
<thead>
<tr>
<th>Want asset at ( T )</th>
<th>Cost at 0</th>
<th>Cost at ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy forward</td>
<td>0</td>
<td>(-F)</td>
</tr>
<tr>
<td>buy spot</td>
<td>(-(S - I))</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
F = (S - I)e^{rT}
\]
Example of an Investment Asset that Provides a Known Income $I$

What is the price of a forward contract to buy a coupon-bearing bond whose current price is $950? The forward contract matures in one year and the bond matures in 3 years. Coupon payments of $10 are expected after 6 months and 12 months (immediately prior to the delivery date of the forward contract). The 6-month and 12-month risk-free interest rates are 5% per annum and 6% per annum.
Investment Asset that Provides a Known Dividend Yield $q$

<table>
<thead>
<tr>
<th>Want asset at $T$</th>
<th>Cost at 0</th>
<th>Cost at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>buy 1 unit forward</td>
<td>0</td>
<td>$-F$</td>
</tr>
<tr>
<td>buy $e^{-qT}$ unit spot</td>
<td>$-Se^{-qT}$</td>
<td>0</td>
</tr>
</tbody>
</table>

$$F = Se^{(r-q)T}$$
Example of an Investment Asset that Provides a Known Dividend Yield $q$

What is the price of a 9-month forward contract on a stock index that is expected to provide a continuous dividend yield of 3% per annum? The risk-free rate of interest is 10% per annum and the stock index price is $250.

What is the value of a long position in this contract originated 2 months ago at a delivery price of $240?
Index Arbitrage

• buy low & sell high
• $F > Se^{(r-q)T}$ → buy spot & short futures
• $F < Se^{(r-q)T}$ → long futures & sell spot
• Simultaneous trades in futures and many different stocks, often effected by program trading
• Sometimes (e.g., October 87), simultaneous trades are not possible.
Short Selling

- Short selling is selling securities you do not own.
- Your broker borrows the securities from another client and sells them in the usual way.
- You must pay dividends & other benefits the owner receives.
- When closing out, you must replace the securities in the account of the original client.

(Hull) You short sell 500 IBM shares at $120, a $1 dividend is paid, and close out at $100. What is your profit?
Repo Rate

• The repo rate is the relevant interest rate for many arbitrageurs.

• A repurchase agreement (repo) is an agreement where one financial institution sells securities to another financial institution and agrees to buy them back later at a slightly higher price.

• The difference between the selling price and the buying price is the interest earned.
Currency Futures

The foreign risk-free interest rate $r_f$ can be viewed as a dividend yield $q$.

\[
\begin{array}{|c|c|c|}
\hline
\text{Want currency at } T & \text{Cost at 0} & \text{Cost at } T \\
\hline
\text{buy 1 unit forward} & 0 & -F \\
\text{buy } e^{-r_f T} \text{ unit spot} & -Se^{-r_f T} & 0 \\
\hline
\end{array}
\]

\[F = Se^{(r-r_f)T}\]
Consumption Asset

- **Convenience yield** $y$: benefit of owning the underlying asset
- **Cost of carry**: $r + u$

\[
F = Se^{(r+u-y)T}
\]
\[
F \leq Se^{(r+u)T}
\]

Storage costs can also be expressed as a known dollar cost $U$ rather than a known yield $u$

\[
F = (S + U)e^{(r-y)T}
\]
\[
F \leq (S + U)e^{rT}
\]
Homework

1. (Hull 5.9) A one-year long forward contract on a nondividend-paying stock is entered into when the stock price is $40 and the risk-free rate of interest is 10 percent per annum with continuous compounding.
   (a) What are the forward price and the initial value of the contract?
   (b) Six months later, the price of the stock is $45 and the risk-free interest rate is still 10 percent. What are the forward price and the value of the forward contract?

2. (Hull 5.11) Assume that the risk-free interest rate is 9 percent per annum with continuous compounding and that the dividend yield on a stock index varies throughout the year. In February, May, August, and November, it is 5 percent per annum. In other months, it is 2 percent per annum. Suppose that the value of the index on July 31 is 300. What is the futures price for a contract deliverable on December 31?

3. (Hull 5.13) Estimate the difference between short-term interest rates in Mexico and the United States on February 4, 2004 from the following information:

<table>
<thead>
<tr>
<th>delivery</th>
<th>settle ($/Peso)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar</td>
<td>.08920</td>
</tr>
<tr>
<td>June</td>
<td>.08812</td>
</tr>
</tbody>
</table>

4. (Baby Hull 5.23, Papa Hull 5.24) A stock is expected to pay a dividend of $1 per share in two months and in five months. The stock price is $50 and the risk-free rate of interest is 8 percent per annum with continuous compounding for all maturities. An investor has just taken a short position in a six-month forward contract on the stock.
   (a) What are the forward price and the initial value of the forward contract?
   (b) Three months later, the price of the stock is $48 and the risk-free rate of interest is still 8 percent per annum. What are the forward price and the value of the short position in the forward contract?

5. (Baby Hull 5.25, Papa Hull 5.26) A company that is uncertain about the exact date when it will pay or receive a foreign currency may try to negotiate with its bank a forward contract that specifies a period during which delivery can be made. The company wants to reserve the right to choose the exact
delivery date to fit in with its own cash flows. Put yourself in the position of the bank. How would you price the product that the company wants?