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Auctioning divisible goods★

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Summary. We derive equilibrium bidding strategies in divisible good auctions for asymmetrically informed risk neutral and risk averse bidders when there is random noncompetitive demand. The equilibrium bid schedules contain both strategic considerations and explicit allowances for the winner's curse. When the bidders' information is symmetric, the strategic aspects of bidding imply that there always exist equilibria of a uniform-price auction with lower expected revenue than provided by a discriminatory auction. When bidders are risk averse, there may exist equilibria of the uniform-price auction that provide higher expected revenue than a discriminatory auction.

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1 Introduction

Many markets exist for which the exchange mechanism is best characterized as an auction for a divisible good. An important example comes from the financial

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markets, the primary market for U.S. government debt. There are many others. While a well developed literature concerning unit-demand auctions has existed since the 1980's, only recently has significant attention been paid to auctions for divisible goods (or shares). Because of the widespread use of this mechanism, it is important that we increase our understanding of these auctions.

In a divisible good auction, a seller offers some amount of a good for sale at auction. Once the bids are submitted, the "stop-out" price (the price at which aggregate demand equals the available supply) and the individual bids determine the allocation of the good to the bidders. Bids submitted at prices greater than or equal to the stop-out price are winning bids. In uniform-price auctions, all winning bids are filled at the stop-out price, while in completely discriminatory auctions, winning bids are filled at the bid price. A distinguishing feature of divisible good auctions is that bidders are allowed to submit multiple price/quantity pairs as bids. These bid schedules specify the quantity (or fraction) of the good desired by the bidder at each price. Much of the difficulty associated with the study of divisible good auctions rests with the fact that this allows the bidders a very large strategy space. While in unit-demand auctions bidders compete only through price, in divisible good auctions bidders may choose among all weakly downward sloping bid schedules. Thus, understanding the strategic nature of bidding in divisible good auctions is an important aspect of the problem.

We present a general, common values, model of Treasury auctions that explicitly accounts for a perfectly divisible good, an arbitrary number of bidders, different levels of price discrimination, and the presence of random noncompetitive demand.[1] The auction rules imply a bidder's expected utility is dependent upon his entire bid schedule, i.e., is a functional, so restricting attention to strategies (bid schedules) that are piecewise continuously differentiable allows standard techniques in dynamic optimization to identify the optimal strategies.

Results are derived for two different informational structures. In order to concentrate on the strategic aspects of bidding in divisible good auctions we consider a case in which the competitive bidders have symmetric information.[2] The resulting equilibrium bid schedules nicely illustrate the strategic aspects of bidding in divisible good auctions. We confirm several intuitive comparative static results concerning the equilibrium bidding.

The central results of the paper are characterizations of equilibria when the competitive bidders possess asymmetric information. These characterizations demonstrate that equilibrium bid schedules take explicit consideration of the "winner's curse" and that (almost) all also contain some amount of strategic bidding.

A specific solution of the general characterizations with asymmetric information in a uniform-price auction is provided. The intuitions from the general

[1] In U.S. Treasury auctions bidders may submit noncompetitive bids that are guaranteed to be filled.

[2] Throughout the paper, we will assume that the seller's information is inferior to that of the bidders. This is a common rationale for the use of an auction. The terms *symmetric* or *asymmetric* information will refer to the information partition of the competitive bidders relative to one another.

model are illustrated by this example and results from the symmetric information case are shown to extend to the case of private bidder information.

Recently, there has been considerable attention paid to divisible good (or multi-unit) auctions;[3] however, our understanding of this important mechanism is far from complete. The most closely related papers are Wilson [30] and Back and Zender [2]. In a model of share auctions, Wilson [30] showed that the selling price can be significantly lower if bidders are allowed to submit bid schedules rather than a single price bid. Back and Zender [2] demonstrates explicitly how a uniform-price auction can yield a variety of results (most of which are inferior for the seller), illustrates the strategic difference between unit-demand auctions and auctions of divisible goods, and compares uniform-price and discriminatory auctions. Mixed empirical results regarding the choice of auction formats provide an additional motive for further theoretical analysis of divisible good auctions.[4]

The paper is organized as follows. Section 2 describes a model of divisible good auctions. Section 3 characterizes symmetric equilibrium bidding strategies when the competitive bidders are symmetrically informed. The case of asymmetric bidder information is analyzed in Section 4. The unique linear solution of a uniform-price auction model with asymmetric bidder information is presented in Section 4.4 to further illustrate the effect of private information. Section 5 concludes. Appendices provide the proofs of our results.

2 The model

Although our results are more generally applicable, we present them in the context of auctions for U.S. Treasury securities. This section describes the model and relates its features to those of the Treasury auction environment.[5]

A risk neutral seller has one unit of a perfectly divisible, risky asset for sale. The seller's information concerning the value of the good is inferior to that of

[3] A partial list of recent work includes the following. Maskin and Riley [18] and Branco [6] study divisible good auctions in a mechanism design framework but do not directly compare uniform-price and discriminatory auctions. In Tenorio [27], a private values model is used to describe Treasury auctions. Bidders are allowed to demand *multiple* units at the *same* price. A three-stage model in Edsparr [9] considers the cross-market interactions surrounding Treasury auctions. Ausubel and Cramton [1] discuss demand reduction in divisible good auctions and its impact on allocational efficiency. Engelbrecht-Wiggans and Kahn [10 and 11] consider auctions in which bidders may bid for two units of a good at different prices. Perry and Reny [22] show that the *linkage principle* does not necessarily hold in multi-unit auctions. Back and Zender [3], Lengwiler [16] and McAdams [20] consider multi-unit auctions with endogenous supply.

[4] For example, Baker [4] and Simon [24] examine the U.S. Treasury's experience with uniform-price auctions in the early 1970's and find that they cost the Treasury money. However, due to sample size and other data restrictions, the authors caution careful interpretations of their results. Umlauf [28] analyzes the Mexican Treasury auctions during 1986-1991, and finds that bidder profits dropped to nearly zero when that Treasury switched from a discriminatory to a uniform-price format in 1990. There is, however, openly acknowledged collusion between the largest bidders in the Mexican auctions, and this and other institutional differences make these results difficult to compare with those from other auctions.

[5] Treasury auctions actually take place in terms of yields, not prices. We present our analysis in terms of prices for simplicity and added generality.

the competitive bidders' and she chooses a divisible good auction as an exchange mechanism because of this informational disadvantage. The seller's objective is to maximize her expected revenue from the auction and her choice variables are the auction's pricing rule and its reserve price.

There are $N > 2$ competitive bidders,[6] each acting to maximize his expected utility. We consider both risk neutral and risk averse competitive bidders. Risk aversion is introduced by assuming the competitive bidders act as if to maximize a derived mean-variance utility of profit function with a common risk aversion parameter, ρ.[7]

Noncompetitive bids are also allowed for in the model. The total noncompetitive demand, which has priority over the competitive bids, is given by the random variable $\tilde{z}$, with support $[0, 1]$. The competitive bidders, therefore, compete for an uncertain quantity, $1 - \tilde{z}$. The pdf and cdf for $\tilde{z}$ are denoted $g(\ )$ and $G(\ )$ respectively.[8]

When an explict distribution for noncompetitive demand is required (the discriminatory auctions in Section 3), we assume that $\tilde{z} \in [0, 1]$ has the probability density function:

$$g(z) = \frac{1}{\theta} z^{\frac{1-\theta}{\theta}}. \tag{1}$$

These are the inverted pareto distributions (with $\theta \in (0, \infty)$ and the β parameter restricted to be one) and are used for analytical tractability.

Competitive bidder i's strategy is a schedule, $x_i(p, s_i)$, that specifies his quantity demanded at different price levels, p, and for different realizations of his private signal $s_i \in S$, where S is a signal space with a possibly infinite number of elements. Bid schedules are assumed to be piece-wise continuously differentiable with respect to price.[9] Only bid schedules that are weakly decreasing in price are considered.

After the individual demands are aggregated, the *stop-out price* for the auction is defined as the highest price, p, at which aggregate excess demand is nonnegative:[10]

[6] In U.S. Treasury auctions primary dealers, large instituitions required to maintain active markets in the auctioned securities, may submit competitive bids.

[7] For uniform-price auctions, this utility function may be derived from the combination of CARA utility and normally distributed random variables. However, in the case of discriminatory auctions, these assumptions do not imply a quadratic objective function in the dynamic optimization problem. This occurs because the pointwise maximization that works with a uniform-price auction does not work in a discriminatory auction. This representation of risk aversion is also used in Biais [5].

[8] In the U.S. Treasury auctions, individuals or institutions may submit noncompetitive (or quantity) bids (for up to $10 million in face value) that are guaranteed to be filled.

[9] While in (the U.S.) treasury auctions competitive bidders are allowed to submit as many price-quantity pairs as they wish, in practice they seem to use a relatively small number (see, for example, Bikhchandani and Huang [7]). The assumption of piece-wise continuous bid schedules is not, therefore, reflective of actual practice. It does, however, allow us to learn a great deal about the problem in a tractable environment. Work in progress examines the issue of equilibrium bidding strategies in a discrete framework.

[10] This definition assumes that the reserve price is zero.

$$p = \begin{cases} \max\{p \mid \sum_i x_i(p, s_i) \geq 1 - z; p \geq 0\} & \text{if } \{p \mid \sum_i x_i(p, s_i) \geq 1 - z; p \geq 0\} \neq \emptyset, \\ 0 & \text{if } \{p \mid \sum_i x_i(p, s_i) \geq 1 - z; p \geq 0\} = \emptyset. \end{cases}$$

The intrinsic value of the asset is determined by the random variable $\tilde{v}$.[11] We assume that the *prior* distribution of asset value is common knowledge.

Prior to the submission of bids, the competitive bidders receive private signals concerning the value of the asset. The competitive bidders' valuation of the asset is, therefore, a conditional valuation. The seller does not observe the signals and so knows only the distribution of the conditional expected value. We will suppress the dependence of the conditional distribution of value on the bidders' signals for ease of notation and everywhere consider $\tilde{v}$ and its distribution from the competitive bidders' point of view. In Section 3 we consider the case in which all competitive bidders receive the same signal (symmetric bidder information) and in Section 4 the case of differential private information on the part of the bidders is examined. With risk neutral competitive bidders, we assume that the conditional expected value is well defined. For risk averse competitive bidders, we will assume that the conditional mean ($\mathrm{E}[\tilde{v}] = \overline{v} \geq 0$) and variance of the asset's value ($\mathrm{Var}[\tilde{v}] = \tau_v^{-1}$) are well defined.

In an auction with a pricing rule given by $\alpha \in [0, 1]$ and a stop-out price p, bidder i's actual profit is:

$$\tilde{\pi}_i(p, s_i) = (\tilde{v} - p)x_i(p, s_i) - \alpha \int_p^{p_{max}} x_i(t, s_i)dt,$$

where p_{max} is the largest price for which demand is nonnegative. The uniform-price ($\alpha = 0$) and the discriminatory auctions ($\alpha = 1$) are special cases of this formulation. The parameter α represents the proportion of the area under a competitive bidder's bid schedule captured by the seller.

The solution concept is Bayesian-Nash equilibrium in bidding strategies. Only symmetric, pure strategy equilibria are considered. A symmetric strategy Bayesian-Nash equilibrium is a set of strategies $x_i(p, s_i) = x(p, s_i)$ for the N competitive bidders such that for each bidder i, $x(p, s_i)$ maximizes his expected utility of profit for all $s_i \in S$.

3 Equilibria with symmetric bidder information

This section of the paper characterizes all equilibria in piece-wise continuous bid schedules when the competitive bidders have symmetric information and there is random noncompetitive demand. Abstracting from issues related to information aggregation, highlights intuitions related to the strategic aspects of bidding in divisible good auctions.[12]

[11] Typically, the competitive bidders are not the ultimate holders of the auctioned securities. It is therefore appropriate to consider the resale price of the securities in the secondary market to be the *common value* of the assets. This after-market price is unknown at the time bids are submitted.

[12] Also, since the U.S. government securities market is characterized by tremendous liquidity and is generally believed to involve small degrees of asymmetric information among the most active traders

3.1 Risk neutral bidders

Uniform-price auctions

Proposition 3.1 *In a uniform-price auction with symmetric bidder information, the equilibrium bid schedules submitted by the N risk neutral competitive bidders satisfy the following condition:*

$$(N-1)x'(p) \quad = \quad -\frac{x(p)}{\bar{v}-p}, \tag{2}$$

which may be solved (in inverse form) as:

$$p(x) \quad = \quad \bar{v}[1-(\frac{x}{x_0})^{N-1}]. \tag{3}$$

Proof. To avoid duplication, we utilize Eq. (38) which is a result proved in Appendix A. In Eq. (38), setting the parameters $\alpha = 0$ (uniform-price auction), $\rho = 0$ (risk neutral bidders), and bidder i's information partition $\{\tilde{\psi}, \tilde{s}_i\} = \tilde{z}$ (no private signal correlated with asset value is available to the bidders), we immediately have Eq. (2).

The solution to Eq. (2) (in inverse form) is:

$$p(x) = \bar{v} - C_1 x^{N-1}.$$

Noticing that the integration constant, C_1, is equal to the quantity at zero price, $x_0 \equiv x(p = 0)$, we can rewrite the solution as Eq. (3). For any $x_0 \in (0, \infty)$, the associated bidding strategy is downward sloping. □

The equilibrium bid schedules in uniform-price auctions (Eq. (3)) are not sensitive to the distributional properties of $\tilde{z}$, the noncompetitive demand. Given the bids submitted by the other competitive bidders, for each realization of $\tilde{z}$, bidder i would not change his bid schedule if allowed to do so *ex post*.[13] The intuition is that, in uniform-price auctions, a bidder's decision to choose his price-quantity combinations is localized in the sense that his optimal response to any particular residual supply curve[14] does not impact his choice with respect to any other realized residual supply curve. If the family of residual supply curves can be parameterized by a single random variable, the set of *ex post* optimal price-quantity pairs may serve as the bidder's *ex ante* optimal bidding strategy.

There exists a continuum of equilibria indexed by the parameter x_0. A very small x_0 implies a very inelastic bid schedule and so a large amount of demand

(Cornell [8] reports that typical bid-ask spreads for inter-dealer trades are about one basis point), the characterization "symmetric bidder information" may therefore represent the Treasury auction environment fairly well.

[13] Solutions possessing such an *ex post* optimality were found by Klemperer and Meyer [14] and Kyle [15].

[14] The residual supply curve facing bidder i is derived by first subtracting the *realized* noncompetitive demand then the equilibrium competitive bids from the available supply. Of course, *ex ante* bidder i does not know which residual supply curve he will face; he has only a conjecture of a family of curves.

reduction. As in Back and Zender [2],[15] the ability to submit (steep) bid schedules in a divisible good auction provides the bidders with an important strategic advantage. At the other extreme, $x_0 = \infty$, each bidder submits a completely elastic (flat) bid schedule at a price of $\overline{v}$.

Mathematically, any x_0 can be part of the solution to the Euler equation presented in Eq. (2). Economically, given the interpretation of x_0, only $x_0 \in (0, \infty)$ are meaningful. The equilibrium bid schedules for $x_0 \in (0, \frac{1}{N})$ allow the possibility of a negative realized stop-out price. For $x_0 \in [\frac{1}{N}, \infty)$, however, no point on the corresponding equilibrium bid schedules that can be "hit" has a price that is less than zero.

To limit the competitive bidders' strategic advantage and avoid "unreasonable" outcomes, yet leave the analysis as general as possible, we assume throughout this section that the seller imposes a reserve price equal to zero, effectively restricting $x_0 \geq \frac{1}{N}$.[16]

Corollary 3.1 *When the N competitive bidders in a uniform-price auction are risk neutral, there exists a continuum of equilibria indexed by x_0. When x_0 is varied from $\frac{1}{N}$ to ∞, the expected stop-out price increases from $\overline{v}(1 - \mathrm{E}[(1 - \tilde{z})^{N-1}])$ to $\overline{v}$. For each given x_0, the equilibrium bid schedules and the expected stop-out price are increasing in $\overline{v}$ and N.*

Proof. The first statement is a direct result of solving a first-order differential equation. Given the equilibrium bids, the realized equilibrium stop-out price behaves as:

$$\tilde{p} = \overline{v}[1 - (\frac{1 - \tilde{z}}{Nx_0})^{N-1}]. \tag{4}$$

The second and third statements follow from an examination of Eqs. (3) and (4). □

Corollary 3.1 indicates that x_0 serves as an index of the amount of strategic bidding or demand reduction in a uniform-price auction. All of the equilibria with strictly downward sloping bid schedules include some demand reduction. Corollary 4.2 establishes that this is a general property of a class of equilibria which encompasses the symmetric information case.

Discriminatory auctions

Proposition 3.2 *In a discriminatory auction with symmetric bidder information, the equilibrium bid schedules submitted by the N risk neutral competitive bidders satisfy the following condition:*

[15] This solution is equivalent to Back and Zender's Theorem 4. Here we have also demonstrated that this set of equilibria represents all possible symmetric equilibria in bid schedules that are piecewise continuously differentiable, and that all such equilibria are *ex post* optimal.

[16] A reserve price greater than zero may improve the seller's expected profits, it however also exposes the seller to the risk of undersubscription in the auction.

$$(N-1)x'(p) \quad = \quad -\frac{(G/g)(1-Nx(p))}{\bar{v}-p}. \tag{5}$$

If z has an inverted pareto distribution ($\beta = 1$), this equation may be solved as:

$$p(x) \quad = \quad \bar{v}[1-(\frac{1-Nx_0}{1-x})^{\frac{N-1}{N\theta}}]. \tag{6}$$

Proof. Eq. (5) is again a special case of Eq. (38), proved in Appendix A, by setting $\alpha = 1$ (discriminatory auction), $\rho = 0$, and bidder i's information partition $\{\tilde{\psi}, \tilde{s}_i\} = \tilde{z}$. □

Corollary 3.2 *When the N competitive bidders are risk neutral, the unique equilibrium with weakly downward sloping bid schedules in a discriminatory auction with a reserve price of zero is:*

$$p(x) = \bar{v}. \tag{7}$$

This is obtained when $x_0 = \frac{1}{N}$. The stop-out price is independent of N and given by $\bar{v}$.

The set of equilibria is dramatically reduced by imposing two restrictions. $x_0 \leq \frac{1}{N}$ ensures the bid schedule is downward sloping. A reserve price of zero implies x_0 is no less than $\frac{1}{N}$.[17]

With risk neutral bidders, discriminatory pricing intensifies bidder competition to the fullest extent, the bidders compete by submitting flat bid schedules.[18] In contrast to the uniform-price auction, the seller's informational disadvantage does not limit the extent to which she can control strategic bidding. A reserve price of zero together with discriminatory pricing eliminates all of the bidders' strategic advantage.

Revenue comparison

The revenue comparison is limited by the dependence of the equilibrium bid schedules, for discriminatory auctions, on the distribution of the noncompetitive demand. It must be acknowledged that the results may not be robust to alternative specifications of this distribution function.

[17] This is the equilibrium identified in Back and Zender [2] for risk neutral bidders in a discriminatory auction. We have extended this result to allow for noncompetitive demand and have shown that this is the unique equilibrium when bid schedules are weakly downward sloping and do not include negative prices.

[18] A related result from Engelbrecht-Wiggans and Kahn [10] shows that when bidders have decreasing marginal values for units of the good, in equilibrium they will submit equal bids for several units with positive probability. Empirically, we would expect bids to be more dispersed in uniform-price auctions as compared with bids in discriminatory auctions if bidders are risk neutral. Reinhart and Belzer [23] and Feldman and Reinhart [12] present results consistent with this prediction.

Proposition 3.3 *When the competitive bidders are risk neutral and symmetrically informed, the seller's expected revenue is $\bar{v}$ in the unique equilibrium of a discriminatory auction. The seller's expected revenue is less than $\bar{v}$ in (almost) all equilibria of a uniform-price auction.*[19]

In a uniform-price auction, the seller's expected revenue is strictly increasing in x_0.

Proof. See Appendix B. □

3.2 Risk averse competitive bidders

Uniform-price auctions

Proposition 3.4 *In a uniform-price auction with symmetric bidder information, the equilibrium bid schedules submitted by the N risk averse competitive bidders satisfy the following condition:*

$$(N-1)x'(p) \quad = \quad -\frac{x(p)}{\bar{v}-p-\rho\tau_v^{-1}x(p)}, \tag{8}$$

which may be solved as:

$$p(x) \quad = \quad \bar{v}[1-(\frac{x}{x_0})^{N-1}] - x[1-(\frac{x}{x_0})^{N-2}]\frac{(N-1)\rho\tau_v^{-1}}{N-2}. \tag{9}$$

Proof. Eq. (8) is a special case of Eq. (38), proved in Appendix A, by setting the parameter $\alpha = 0$, and bidder i's information partition $\{\tilde{\psi}, \tilde{s}_i\} = \tilde{z}$. □

With risk averse bidders, there also exists a continuum of equilibria in a uniform-price auction. It can be shown that, for $p(x)$ to be weakly downward sloping ($\forall x \in [0, x_0]$), x_0 must be no greater than $\bar{v}/(\rho\tau_v^{-1})$, a restriction that implies all equilibrium bid schedules are strictly downward sloping. A reserve price of zero restricts x_0 to be at least $\frac{1}{N}$.[20]

Corollary 3.3 *Assume that $\frac{1}{N} \leq \frac{\bar{v}}{\rho\tau_v^{-1}}$. When the N competitive bidders in a uniform-price auction are risk averse, there exists a continuum of equilibria indexed by x_0. When x_0 is varied from $\frac{1}{N}$ to $\frac{\bar{v}}{\rho\tau_v^{-1}}$ the expected stop-out price increases. For each x_0, the equilibrium bid schedules and the expected stop-out price are decreasing in ρ, and are increasing in $\bar{v}$, N, and τ_v.*[21]

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[19] The only exception is that, for $x_0 = \infty$, the expected revenue from a uniform-price auction is equal to $\bar{v}$.

[20] The two restrictions on x_0 require that $\frac{1}{N} \leq \bar{v}/(\rho\tau_v^{-1})$, or $\rho \leq N\bar{v}/\tau_v^{-1}$. Given that τ_v^{-1} is much smaller than $N\bar{v}$ in Treasury auctions and that the competitive bidders are large institutions that are close to being risk neutral, this implicit upper bound on ρ is, therefore, not expected to bind.

[21] The fact that the equilibrium bid schedules are increasing in τ_v implies that it is always in the seller's interest to promote the release of information. This is due to the presence of risk aversion and demand reduction and should not be confused with a similar conclusion from the unit auction literature related to the winner's curse. Directly related to the unit auction result, Corollary 4.5 shows that when the competitive bidders in a divisible good auction possess asymmetric information this same conclusion can be drawn because of the benefit the seller obtains from any reduction in the winner's curse.

Proof. Equation (9) can also be written as:

$$p(x) = \bar{v} - \frac{\rho\tau_v^{-1}(N-1)}{N-2}x - C_1 x^{N-1},$$

where

$$C_1 = (\bar{v} - \frac{N-1}{N-2}\rho\tau_v^{-1}x_0)\frac{1}{x_0^{N-1}}.$$

C_1 is strictly decreasing in x_0 if and only if $x_0 \leq \frac{\bar{v}}{\rho\tau_v^{-1}}$. Therefore, as x_0 increases from $\frac{1}{N}$ to $\frac{\bar{v}}{\rho\tau_v^{-1}}$, the expected stop-out price (which is decreasing in C_1) increases. The second statement follows from substituting $x = \frac{1-z}{N}$ into Eq. (9), considering the corresponding values of x_0 and taking its expectation. The rest of the proof is similar to that of Corollary 3.1. □

For $x_0 \leq \bar{v}/(\rho\tau_v^{-1})$, the expected stop-out price and the seller's expected revenue are increasing in x_0. The competitive bidders' certainty equivalent utility is strictly positive in all of the equilibria and decreasing in x_0. x_0 again serves as an index of the amount of strategic bidding associated with the equilibria of the uniform-price auction.

Examples

Quite generally, a large variety of bid schedules can support Nash equilibria in a uniform-price auction if there is no supply noise. The presence of stochastic noncompetitive demand proves to be very useful in choosing among the plethora of Nash equilibria. As one example of the infinite number of equilibria possible in the absence of supply noise, consider:

$$p(x) = \bar{v} + \rho\tau_v^{-1}x - \frac{\rho\tau_v^{-1}}{2} - C_1\sqrt{\frac{x}{1-x}},$$

a solution constructed by Wilson [30] for a (uniform-price) share auction with two bidders competing for one unit of a good.[22] In this example, the sale price can be made arbitrarily small when C_1 is made large enough.

In contrast, the solution to Eq. (8) in a similar context is:[23]

$$p(x) = \bar{v} + \rho\tau_v^{-1}x\ln x - C_1 x.$$

Notice that while our solution solves Wilson's problem, the reverse is not true. The difference is that our solution is robust with respect to the presence of uncertain noncompetitive demand, whereas Wilson's solution is an equilibrium bidding strategy only if both bidders know, *ex ante*, their equilibrium quantity allocation.

[22] Both bidders are assumed to possess CARA utility. Asset value is normally distributed. To ensure that the bid schedule is downward sloping, C_1 must be greater than $\rho\tau_v^{-1}3\sqrt{3}/8$.

[23] Here, C_1 must be greater than $\rho\tau_v^{-1}$ for the bid schedule to be downward sloping.

In other words, without supply uncertainty, an equilibrium bid schedule only needs to be an optimal response (to the schedules submitted by the other bidders) at a single point. With supply uncertainty this condition must be satisfied at all points that have a positive probability of being "hit."

As a further illustration of this point, consider Wilson's example when the bidders know the value of the good with certainty. This corresponds with our case of risk neutral competitive bidders with symmetric information. For a uniform-price auction with N bidders his solution is a linear bid schedule:

$$p(x) = \frac{N\bar{v}}{2} - \frac{N\bar{v}}{2}(N-1)x. \tag{10}$$

Our solution in this case is:

$$p(x) = \bar{v}[1 - (\frac{x}{x_0})^{N-1}], \qquad x_0 \in [1/N, \infty). \tag{11}$$

While any of our bid schedules solve Wilson's problem the reverse is not true. With risk averse bidders there is a downward sloping linear equilibrium that solves Eq. (9), with risk neutral bidders the only "linear" solution robust to uncertain supply is $p(x) = \bar{v}$. Eq. (10)'s lack of robustness to supply noise is dramatically illustrated by noting that, for small x it includes bids at prices far above the value of the good.

Discriminatory auctions

Proposition 3.5 *In a discriminatory auction with symmetric bidder information, the equilibrium bid schedules submitted by the N risk averse competitive bidders satisfy the following condition:*

$$(N-1)x'(p) \quad = \quad -\frac{(G/g)(1-Nx(p))}{\bar{v} - p - \rho\tau_v^{-1}x(p)}. \tag{12}$$

If z has an inverted pareto distribution ($\beta = 1$), this equation may be solved as:

$$\begin{aligned} p(x) \quad &= \quad \bar{v} - \frac{\rho\tau_v^{-1}}{N(\theta+1)-1}[\theta + (N-1)x] \\ &\quad -(\frac{1-Nx_0}{1-Nx})^{\frac{N-1}{N\theta}}\{\bar{v} - \frac{\rho\tau_v^{-1}}{N(\theta+1)-1}[\theta+(N-1)x_0]\}. \end{aligned} \tag{13}$$

Proof. Eq. (12) is a special case of Eq. (38), proved in Appendix A, by setting the parameter $\alpha = 1$, and bidder i's information partition $\{\tilde{\psi}, \tilde{s}_i\} = \tilde{z}$. □

Corollary 3.4 *For N risk averse competitive bidders in a discriminatory auction with a reserve price of zero the unique weakly downward sloping equilibrium bid schedule is:*

$$p(x) = \bar{v} - \frac{\rho\tau_v^{-1}}{N(\theta+1)-1}[\theta + (N-1)x]. \tag{14}$$

This is obtained when $x_0 = \frac{1}{N}$. *The expected stop-out price*

$$\mathrm{E}[\tilde{p}] \quad = \quad \bar{v} - \frac{\rho \tau_v^{-1} \theta [N(\theta + 2) - 1]}{N(\theta + 1)[N(\theta + 1) - 1]}$$

increases in $\bar{v}$, N, *and the mean noncompetitive demand,* $\bar{z} = \frac{1}{1+\theta}$, *and decreases in* τ_v^{-1} *and* ρ.

Proof. There exists a continuum of equilibrium. However, restricting to downward sloping bid schedules and imposing a reserve price of zero, only the case of $x_0 = \frac{1}{N}$ remains. The comparative statics obtained are similar to those in uniform-price auctions. □

The existence of a continuum of equilibria in discriminatory auctions and the elimination of all but one of these equilibria by the use of a reserve price is related to conflicting statements in Back and Zender [2] and Wilson [30]. Wilson states that for any equilibrium bidding strategy, $p(x)$, in a uniform-price auction there is a corresponding equilibrium in a discriminatory auction, given by $q(x) = p(x) + xp'(x)$, providing the same expected revenue. This claim is often interpreted as a revenue equivalence result. Theorems 1 and 3 in Back and Zender show, for the case of no private bidder information with no noncompetitive demand, that while there are equilibria in uniform price auctions providing the seller with any level of revenue in the range $[0, \bar{v}Q]$, the seller's revenue is $\bar{v}Q$ in any pure-strategy equilibrium of a discriminatory auction.[24]

Our results relate to this issue in two ways. First, the bid schedules derived for uniform-price and discriminatory auctions are the only equilibria (for the class of piece-wise continuously differentiable bid schedules), therefore, there are no equilibrium bid schedules for uniform-price auctions related to equilibrium bid schedules in discriminatory auctions according to Wilson's equation. Second, the minimal restriction that no bids at negative prices are accepted, implies there are multiple equilibria in uniform-price auctions (with differing expected revenue) but only one equilibrium for the discriminatory auction.

Examination of Eq. (14) indicates that the certainty equivalent utility of a risk averse competitive bidder in the only equilibrium of a discriminatory auction is strictly positive so the equilibrium bid schedules include some demand reduction. Risk averse bidders do not compete as aggressively as risk neutral bidders so some strategic bidding survives in equilibrium. Nevertheless, numerical calculations show that the certainty equivalent utility of risk averse bidders is larger in *all* equilibria of the uniform-price auction than in the only equilibrium of the discriminatory auction (see Appendix B). Price discrimination is therefore effective in limiting the strategic advantage possessed by risk averse *competitive* bidders.

[24] The equilibrium requirement does not appear to be met for Wilson's proposed bids in the discriminatory auction.

Revenue comparison

Proposition 3.6 *When the competitive bidders are risk averse and symmetrically informed, the seller's expected revenue in a discriminatory auction is strictly greater than the expected revenue from the bidders' most preferred equilibrium* $(x_0 = \frac{1}{N})$ *of a uniform-price auction.*

In a uniform-price auction, the expected revenue is increasing in x_0*. If* $\bar{z}$ *and* N *are sufficiently large,*[25] *there exist equilibria* (x_0 *near* $\frac{\bar{v}}{\rho \tau_v^{-1}}$) *of the uniform-price auction that generate a larger expected revenue than does a discriminatory auction.*[26]

Proof. See Appendix B. □

The competitive bidders' expected profit and certainty equivalent utility are always (i.e., for all x_0) higher in a uniform-price auction than in a discriminatory auction. The final statement in Proposition 3.6 is explained by the fact that, the expected profit of the noncompetitive bidders is larger in discriminatory auctions than it is in uniform-price auctions when N and $\bar{z}$ are large. When competitive bidders in a uniform-price auction, for reasons outside of the current model, do not select a relatively advantageous equilibrium[27] (and N and $\bar{z}$ are sufficiently large), this becomes the dominant effect.

Corollary 3.5 *Under both pricing rules the seller's expected revenue is increasing in* $\bar{v}$, N, *and* $\frac{1}{1+\theta}$ *and decreasing in* ρ *and* τ_v^{-1}.

Proof. See Appendix B. □

The obvious subset of comparative static results holds in the risk neutral case.

4 Equilibria with asymmetric bidder information

The most general results of this paper are characterizations of equilibrium bidding strategies in divisible good auctions when risk averse or risk neutral bidders possess asymmetric information. In general, deriving the equilibrium strategies in a divisible good auction is difficult with asymmetric bidder information due to the inference problem equilibrium bids must address combined with the strategic component of bidding. In order to formulate the problem in a way that will allow the use of control theory the inference problem must be simplified, requiring that we consider only a subset of the equilibria.

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25 We can compute numerically the precise indifference curve that separates parameter regions where the seller prefers one auction format versus another.

26 Experimental evidence (see Smith [25]) is consistent with the results of Propositions 3.3 and 3.6.

27 For example, Back and Zender [3] show that if the seller retains the right to restrict the supply after all bids are submitted, the bidders are not able to sustain their more preferred equilibria; x_0 has a lower bound that is much larger than $\frac{1}{N}$. Alternatively, the seller may use a reserve price greater than zero. This, however, exposes the seller to the risk of under-subscription.

We restrict attention to equilibria in bid schedules that are additively separable in price and a bidder's private signal, $x(p, \tilde{s}_i) = x_1(p) + x_2(\tilde{s}_i)$. In other words, if bidder i expects the other bidders to submit separable schedules, his best response is also separable in price and signal. This simplifies the problem since, in formulating his optimal bid schedule, if an individual bidder conjectures that the other bidders submit bids that are additively separable, the stop-out price p may be inverted from the market clearing condition. This invertibility implies the existence of a sufficient statistic, $\tilde{\psi}$, summarizing any information regarding $\tilde{z}$ and the $\tilde{s}_j$'s relevant for determining the residual supply curve facing bidder i, such that we can write $p = p(\tilde{\psi})$.[28] The economic meaning of this restriction is that each bidder faces a family of parallel residual supply curves that can be ordered by a single index variable. The optimization technique allows the bidder to then choose among all weakly downward sloping piece-wise continuously differentiable bid schedules.

4.1 Risk neutral competitive bidders

We first characterize the signal-dependent bidding strategies of risk neutral competitive bidders assuming that the asset's expected value conditional on the private signal of each bidder is well defined.

Uniform-price auctions

The following proposition characterizes the equilibrium bid schedules for risk neutral bidders in a uniform-price auction.

Proposition 4.1 *If the N competitive bidders are risk neutral, the following condition characterizes any Bayesian-Nash equilibrium of a uniform-price auction in which the privately informed bidders submit symmetric bid schedules that are piecewise continuously differentiable and additively separable:*

$$\sum_{j \neq i} x'(p, s_j) = -\frac{x(p, s_i)}{V(\psi, s_i) - p}, \tag{15}$$

where $\tilde{\psi}$ is a sufficient statistic for $\tilde{z}$ and $\tilde{s}_j$, $(j = 1, 2, \ldots, N, j \neq i)$, in determining bidder i's residual supply curve, and $V(\psi, s_i) = \mathrm{E}[\tilde{v} | \psi, s_i]$.

Proof. See Appendix A. □

Discriminatory auctions

The equilibrium solution for risk neutral bidders in a discriminatory auction is characterized as follows.

[28] We can define $\tilde{\psi} = \tilde{z} + \sum_{j \neq i} x_2(\tilde{s}_j)$.

Proposition 4.2 *If the N competitive bidders are risk neutral, the following condition characterizes any Bayesian-Nash equilibrium of a discriminatory auction in which the privately informed bidders submit symmetric bid schedules that are piecewise continuously differentiable and additively separable:*

$$\sum_{j \neq i} x'(p, s_j) = -\frac{(G/g)(\psi)}{V(\psi, s_i) - p}, \tag{16}$$

where $\tilde{\psi}$ is a sufficient statistic for $\tilde{z}$ and $\tilde{s}_j$, $(j = 1, 2, ..., N, j \neq i)$, in determining bidder i's residual supply curve, $g(\ )$ and $G(\ )$ are the pdf and cdf for $\tilde{\psi}$, and $V(\psi, s_i) = \mathrm{E}[\tilde{v}|\psi, s_i]$.

Proof. See Appendix A. □

4.2 Risk averse competitive bidders

To consider risk averse competitive bidders we assume they act as if to maximize a derived utility function defined on the conditional mean and variance of profit and that the asset's expected value and variance, conditional on the private signal of each bidder, are well defined.

Uniform-price auctions

For the case of risk averse bidders in uniform-price auctions Proposition 4.3 characterizes the equilibrium solution.

Proposition 4.3 *If the N competitive bidders are risk averse the following condition characterizes any Bayesian-Nash equilibrium of a uniform-price auction in which the privately informed bidders submit symmetric bid schedules that are piecewise continuously differentiable and additively separable:*

$$\sum_{j \neq i} x'(p, s_j) = -\frac{x(p, s_i)}{V(\psi, s_i) - p - \rho T^{-1}(\psi, s_i) x(p, s_i)}, \tag{17}$$

where $\tilde{\psi}$ is a sufficient statistic for $\tilde{z}$ and $\tilde{s}_j$, $(j = 1, 2, ..., N, j \neq i)$, in determining bidder i's residual supply curve, and $V(\psi, s_i) = \mathrm{E}[\tilde{v}|\psi, s_i]$, $T(\psi, s_i) = \frac{1}{\mathrm{Var}[\tilde{v}|\psi, s_i]}$.

Proof. See Appendix A. □

From Propositions 4.1 and 4.3 we notice that the equilibrium bid schedules in uniform-price ($\alpha = 0$) auctions are not sensitive to the distributional properties of $\tilde{\psi}$, indicating that the solutions for the uniform-price auction are again *ex post* optimal.

Discriminatory auctions

For risk averse bidders in a discriminatory auction the equilibrium solution is characterized in Proposition 4.4.

Proposition 4.4 *If the N competitive bidders are risk averse the following condition characterizes any Bayesian-Nash equilibrium of a discriminatory auction in which the privately informed bidders submit symmetric bid schedules that are piecewise continuously differentiable and additively separable:*

$$\sum_{j \neq i} x'(p, s_j) = -\frac{(G/g)(\psi)}{V(\psi, s_i) - p - \rho T^{-1}(\psi, s_i) x(p, s_i)}, \tag{18}$$

where $\tilde{\psi}$ is a sufficient statistic for $\tilde{z}$ and $\tilde{s}_j$, $(j = 1, 2, \ldots, N, j \neq i)$, in determining bidder i's residual supply curve, $g(\ )$ and $G(\ )$ are the pdf and cdf for $\tilde{\psi}$, and $V(\psi, s_i) = \mathrm{E}[\tilde{v}|\psi, s_i]$, $T(\psi, s_i) = \frac{1}{\mathrm{Var}[\tilde{v}|\psi, s_i]}$.

Proof. See Appendix A. □

The results of Propositions 4.1 through 4.4 can be seen as generalizations of Propositions 3.1, 3.2, 3.4 and 3.5. For example, Eq. (17) reduces to Eq. (8) when the competitive bidders' private signals become valueless so their strategies are not signal-dependent and $\tilde{\psi}$ is informationally equivalent to $\tilde{z}$. This indicates that the strategic aspects of bidding remain when asymmetric information is introduced.

4.3 Properties of the equilibrium solution

Bids in unit demand auctions can be viewed as perfectly elastic bid schedules in the context of divisible good auctions. Therefore, a natural question is whether the preceding formulation of divisible good auctions admits an equilibrium in flat bid schedules. Corollary 4.1 establishes that, with risk neutral bidders, there exists such an equilibrium and that the bidders reveal their true demands. Since risk aversion implies diminishing marginal valuation, the equilibrium bid schedules for risk averse bidders are necessarily downward sloping.

Corollary 4.1 *There does not exist a Bayesian-Nash equilibrium in which risk averse, privately informed bidders submit symmetric bid schedules that are completely flat.*

For risk neutral bidders, it is a Bayesian-Nash equilibrium for the privately informed competitive bidders to submit a flat bid schedule implied by $p = V(p, s_i)$.

Proof. See Appendix C. □

If there is a solution to the equation $p = V(p, s_i)$ it represents an equilibrium bidding strategy in a first-price unit auction.[29]

[29] An adaptation of Theorem 2 in Back and Zender [2] for the presence of noncompetitive demand can also be used to show that, for the case of risk neutral competitive bidders in a discriminatory auction, it is an equilibrium for the competitive bidders to submit bids for the entire quantity at this price.

Explicit solutions to the ODE established in Appendix A are generally difficult to obtain. To better understand the equilibrium bidding behavior of the competitive bidders, we recast the most general characterization of the solution as follows:

$$p = V(\psi, s_i) - \rho T^{-1}(\psi, s_i) x(p, s_i) + \frac{(1-\alpha)x(p, s_i) + \alpha(G/g)(\psi)}{\sum_{j \neq i} x'(p, s_j)}. \quad (19)$$

Given our representation of risk aversion, the first two terms on the right-hand-side are the bidders' marginal valuations.

From Eq. (19), it is clear that whenever the equilibrium bid schedules are strictly downward sloping the third term, which has the interpretation of *bid shading* or *demand reduction*, is always nonpositive.[30] Thus, except for those with completely elastic bid schedules, demand reduction (relative to the marginal valuation curve) is present in all *equilibrium* bid schedules of divisible good auctions regardless of the extent of price discrimination, the degree of informational asymmetry, or the nature of noncompetitive bids.

Corollary 4.2 *Under the assumptions of Propositions 4.1 through 4.4, the equilibrium bid schedules in divisible good auctions almost always involve demand reduction relative to the competitive bidders' marginal valuations. Demand reduction is absent only in the equilibria in which risk neutral bidders submit bid schedules that are perfectly elastic.*

In divisible good auctions, risk aversion affects the bidders' strategies in two ways. First, there is a standard risk premium adjustment to the bidders' marginal valuation functions and so to their equilibrium bid schedules (the second term in Eq. (19)). The third term in Eq. (19), representing the competitive bidders' market power, reinforces this effect and the equilibrium schedules become even more inelastic. Increasing risk aversion results in less aggressive competition and so allows more demand reduction in equilibrium.[31] This is very different from the impact of risk aversion in, for example, a first-price unit auction. In a first-price auction, the effect of increasing risk aversion is to move the equilibrium bids closer to the bidder's valuation of the good.[32] Intuitively, the difference derives from the fact that in a unit auction the risk involved is the difference in payoff from winning or losing the auction. In a divisible good auction, the risk concerns how much of the good a bidder wins.

The inclusion of $V(\psi, s_i)$, the conditional expected value of the asset, in Eqs. (15) through (18) indicates that the equilibrium bid schedules in divisible good auctions with asymmetric information incorporate the standard notion of the

[30] As in Kyle [15], the notion of demand reduction here arises out of the imperfect, oligopolistic competition among the bidders. Note that the size of this term is reduced as the number of competitive bidders gets large. Demand reduction is the focus of a recent paper by Ausubel and Cramton [1]. They show that the demand reduction implies that the common auction forms do not lead to *ex post* efficient allocations.

[31] Tenorio [26] obtains a related result assuming bidders believe their bidding will not affect the outcome.

[32] See, for example, Matthews [19] and Maskin and Riley [17].

winner's curse.[33] In these equations, V is a function of $\tilde{\psi}$ and $\tilde{s}_i$. Nevertheless, the fact that $\tilde{\psi}$ is a sufficient statistic for $\tilde{z}$ and $\tilde{s}_j$ $(j \neq i)$ (in determining bidder i's residual supply curve) implies that $(\tilde{\psi}, \tilde{s}_i)$ is informationally equivalent to $(\tilde{p}, \tilde{s}_i)$. With risk averse bidders, the bids depend upon the expectation of the value of the good, given that the price, p, was the stop-out price, as well as the variance of the value of the good conditional on this same information, extending the standard result.

Because of the complex inference problem embedded in the derivation of the equilibrium bid schedules, obtaining explicit solutions to the general problem is nontrivial. Analytically, we must compute V and T^{-1} conditional on $(\tilde{\psi}, \tilde{s}_i)$ and, *simultaneously,* find a solution to the Euler equation that is consistent with the choice of such a $\tilde{\psi}$. Disentangling the interplay among private information, uncertain noncompetitive demand, price discrimination, and demand reduction, is difficult. To gain insights into the informational properties of divisible good auctions, Section 4.4 presents a parametric example of the preceding formulation for a uniform-price auction with asymmetrically informed risk averse competitive bidders.

4.4 An example of asymmetric information in a uniform-price auction

To further examine the impact of asymmetric information on the equilibrium bidding strategies in a divisible good auction this section examines a special case of our general results. Here, all bidders have CARA utility with the risk aversion coefficient ρ. Bidder i's private signal takes the form: $\tilde{s}_i = \tilde{v} + \tilde{\epsilon}_i$. The signal errors are assumed to be identically distributed: $\tilde{\epsilon}_i \sim N(0, \tau_e^{-1}),\ i = 1, 2, ..., N$.

In equilibrium, each bidder's bid schedule must solve a statistical inference problem, i.e., the bid schedule submitted by each bidder optimally takes into account the effect of his own bids, as well as that of the other bidders and the supply uncertainty, on the auction outcome. A high stop-out price could be the result of aggressive bidding by other informed bidders or high demand by the noncompetitive bidders. These aspects of the model are absent when there is symmetric bidder information.

We assume that the total supply is $Q = 1 + f$ and the noncompetitive demand is $f + \tilde{z}$, where $f \in (0, 1),\ \tilde{z} \sim N(0, \tau_z^{-1})$ and $\tau_z^{-1/2} \ll f$.[34] All primitive random variables of the model are mutually independent.

Proposition 4.5 *In a uniform-price auction with N competitive bidders there exists a unique symmetric equilibrium in linear strategies. This equilibrium is characterized by:*

$$\tilde{x}_i = \mu + \beta\tilde{s}_i - \gamma\tilde{p}, \quad i = 1, 2, \ldots, N, \tag{20}$$

[33] See, for example, Milgrom and Weber [21].

[34] Normality is needed in conjunction with the assumption of CARA utility for tractability reasons. This assumption assigns nonzero probability to the event that noncompetitive bidders sell to the Treasury. The impact of such undesirable events on the equilibrium properties of the model is minimized by focusing on supply shocks that are small relative to the size of the offering.

where β is the unique positive root of a cubic equation:

$$\rho\tau_e^{-1}\beta^3 + \frac{N}{N-1}\beta^2 + \frac{\rho\tau_z^{-1}}{N-1}\beta - \frac{(N-2)\tau_e\tau_z^{-1}}{(N-1)^2} = 0, \tag{21}$$

and

$$\gamma = \beta[1 + \frac{\tau_v\tau_e^{-1}}{1+(N-1)\phi}], \tag{22}$$

$$\mu = (\gamma - \beta)\bar{v} + \frac{\phi}{1+(N-1)\phi}, \tag{23}$$

$$\phi = \frac{(N-1)\beta^2}{(N-1)\beta^2 + \tau_e\tau_z^{-1}} \in (0, \frac{N-2}{2(N-1)}). \tag{24}$$

Proof. See Appendix D. □

Neither linearity nor separability in price and signal represents a restriction on the allowable strategies in Proposition 4.5. In response to linear and separable strategies by the other bidders, each bidder *optimally chooses* to submit a bid schedule that is also linear and separable in price and his own private signal.

In these bidding strategies, γ is the slope of each bidder's bid schedule. It measures how elastic are the equilibrium bid schedules to the stop-out price. β represents each bidder's sensitivity (or responsiveness) to his own private signal. In Appendix D, it is shown that the equilibrium bid schedules are downward sloping and that β is positive, indicating that the bidders respond positively to their private signals. In the limit, as τ_e^{-1} approaches infinity, these equilibrium bid schedules become the linear equilibrium bid schedules contained in the symmetric information case. Confirming the result from the general model, the equilibrium bid schedules become more inelastic when the bidders are more risk averse.

Contrary to the unit auction case but consistent with Ausubel and Cramton [1], in this example, only a partial *allocational efficiency* result is obtained:

Corollary 4.3 *For any realized stop-out price, the competitive bidder who values the good the most receives the largest share.*

Proof. This follows immediately from Proposition 4.5 since β is strictly positive. □

The stop-out price reflects information about the fundamental value of the asset because the competitive bidders respond to their private signals. From an individual bidder's point of view, the variable ϕ measures the efficiency with which the equilibrium stop-out price aggregates the other $N-1$ bidders' private information. Appendix D demonstrates that although ϕ may theoretically lie anywhere between 0 and 1, it will never exceed $1/2$. It is often notationally convenient to use $I \in (1, N-1)$, a monotonic transformation of ϕ, derived in Appendix D with the same interpretation and comparative statics.

Properties of the equilibrium when the competitive bidders submit the symmetric, linear strategies of Proposition 4.5 are described below.

Corollary 4.4 *For the equilibrium in Proposition 4.5, the total expected profits from competitive and noncompetitive bidding are, respectively:*

$$\begin{aligned} N\pi &= \frac{1 + I\tau_z^{-1}}{N\gamma I}, \\ \pi_{nc} &= \frac{f - I\tau_z^{-1}}{N\gamma I}. \end{aligned}$$

The seller's expected revenue is:

$$\begin{aligned} R &= (1+f)\mathrm{E}[\tilde{p}] \\ &= (1+f)(\overline{v} - \frac{1}{N\gamma I}). \end{aligned} \tag{25}$$

In addition, the ex ante variance of each competitive bidder's quantity allocation, and of the stop-out price, are given by:

$$\begin{aligned} \mathrm{Var}[\tilde{x}] &= \frac{\tau_z^{-1} + N(N-1)\beta^2\tau_e^{-1}}{N^2} = \frac{I\tau_z^{-1}}{N^2}, \\ \mathrm{Var}[\tilde{p}] &= \frac{\tau_z^{-1} + N\beta^2\tau_e^{-1} + N^2\beta^2\tau_v^{-1}}{N^2\gamma^2}. \end{aligned}$$

Proof. See Appendix E. □

It is easy to verify that in equilibrium, the *ex ante* expected quantity allocation for each competitive bidder is one N-th of the average quantity available for competitive bidding.

It is evident from Corollary 4.4 that, *ceteris paribus*, the seller prefers higher aggregate demand elasticity ($N\gamma$). This point underlies the intuition that inelastic bid schedules support "collusive seeming" equilibria. Corollary 4.4 also indicates that, *ceteris paribus*, the seller benefits from greater price efficiency (I). More information revelation reduces the severity of the winner's curse leading to a higher level of expected revenue. Thus, there are two forces acting on the seller's expected revenue: the bidders' use of their strategic advantage, and the extent to which the bidders' private information is revealed by the stop-out price.

Finally, consider Eq. (22) in Proposition 4.5 which can be rewritten as follows:

$$\frac{\gamma(\tau_v, \tau_e)}{\tau_v + [1 + (N-1)\phi(\tau_e)]\tau_e} = \frac{\beta(\tau_e)}{[1 + (N-1)\phi(\tau_e)]\tau_e}. \tag{26}$$

Here τ_v measures the precision of public information and $[1 + (N-1)\phi]\tau_e$ can be understood as measuring the *overall* quality of private information in the auction (the first term comes from each bidder's own signal and the second from the signals of the other $N-1$ bidders as reflected in the stop-out price). Eq. (26) suggests that, in equilibrium, the competitive bidders respond to private observations and price signals in proportion to their respective qualities. An immediate implication of this discussion is that the seller should always adopt policies that promote the production and release of public information.

Corollary 4.5 *The seller's expected revenue, R, is strictly increasing in τ_v, the precision of the public information.*

Proof. The result follows from two observations. First, the precision of public information, τ_v, has no effect on the aggregation of private information (β, and, therefore ϕ, are both independent of τ_v). Second, higher quality public information always leads the bidders to submit more elastic bid schedules (γ is strictly increasing in τ_v) and reduces the use of the bidders' strategic advantage in divisible good auctions. □

Corollary 4.5 confirms that a well-established intuition from the unit-demand auction literature (see, e.g., Milgrom and Weber [21]) generalizes to divisible good auctions — namely, that mitigating the winner's curse problem increases the seller's expected revenue. Results from Milgrom and Weber [24, Theorems 20 and 21] and the example in Perry and Reny [22] suggest that CARA utility may play an important role in this result.

5 Concluding remarks

This paper investigates auctions of divisible goods. The Treasury auction environment is modeled, considering the divisible nature of the good offered for sale, the presence of noncompetitive bidding, and different degrees of price discrimination.

Characterizations were provided of equilibria of auctions in which the bidders possess private information. The results show that equilibrium bid schedules in divisible good auctions contain strategic aspects and take explicit account of the "winner's curse."

In the case of symmetric bidder information it was shown that a continuum of equilibria exists for both uniform-price and discriminatory auctions, and that the use of a zero reserve price eliminates all but one of those equilibria in discriminatory auctions. Explicit solutions for the equilibrium bid schedules were provided and their properties were examined.

We also provided an explicit solution for a uniform-price auction with asymmetrically informed bidders. The example illustrates results from the general analysis, the symmetric information case, and from the unit auction literature.

We consider a stand-alone auction. In U.S. Treasury auctions, the potentially complex interplay between the when-issued market, the auction, and the after market suggests that analysis of a stand-alone auction cannot lead to a complete understanding of this important auction environment.[35] Further work on such issues is necessary.

[35] Viswanathan and Wang [29] and Reinhart and Belzer [23] provide theory and empirical evidence, respectively, that suggests this is indeed the case.

A Proof of an ODE for divisible good auctions

Formally, the proof presented below is for the risk averse case. The proof for the risk neutral case follows essentially the same line of reasoning and can be traced back by setting the risk aversion coefficient $\rho = 0$ everywhere in this appendix.

Suppose all bidders j use a certain bid schedule $x(p, \tilde{s}_j)$. As usual, the j subscript runs from 1 to N, and $j \neq i$. Bidder i's problem is to find a bid schedule y that is measurable with respect to the stop-out price, p, and bidder i's own private signal, s_i, such that the conditional expected utility function:

$$\mathrm{E}_{(\tilde{v},\tilde{z},\tilde{s}_j)|s_i}[U(\tilde{\pi}_i)] = \mathrm{E}_{(\tilde{z},\tilde{s}_j)}\mathrm{E}_{\tilde{v}|(z,s_j,s_i)}[U(\tilde{\pi}_i)],$$

is maximized, where $U(\;)$ is a utility function and $\tilde{\pi}_i$ is the bidding profit defined below. With a price discrimination parameter $\alpha \in [0, 1]$, bidder i's uncertain bidding profit is:

$$\tilde{\pi}_i \quad = \quad (\tilde{v} - p)y(p, s_i) - \alpha \int_p^{p_{max}} y(t, s_i)dt,$$

where p_{max} is the intercept of the bid schedule with the price axis.[36] Define the area under the bid schedule as $w(p, s_i) \equiv \int_0^p y(t, s_i)dt$. We can use a change of variable,[37] $y(p, s_i) = w'(p, s_i)$, to rewrite the bidding profit as:

$$\tilde{\pi}_i \quad = \quad (\tilde{v} - p)w'(p, s_i) - \alpha[w(p_{max}, s_i) - w(p, s_i)], \tag{27}$$

The stop-out price, p, in the preceding formulation is itself a random variable determined by the market clearing condition. We restrict attention to conjectures made by bidder i concerning the bids submitted by the $N - 1$ other competitive bidders such that under this conjecture p is invertible from the above market clearing condition in the sense that there exists a sufficient statistic, $\tilde{\psi}$, which summarizes all relevant information regarding $\tilde{z}$ and $\tilde{s}_j$ in determining bidder i's residual supply curve, such that we can write $p = p(\tilde{\psi})$.

Denoting the pdf of $\tilde{\psi}$ conditional on s_i as $g(\;)$ [the corresponding cdf is denoted $G(\;)$], we assume that bidder i chooses his optimal bidding function to maximize the following derived mean-variance utility function:

$$\begin{aligned} & \mathrm{E}_{\tilde{\psi}}[(V(\psi, s_i) - p)w'(p, s_i) \\ & -\frac{\rho}{2T(\psi, s_i)}w'^2(p, s_i) + \alpha w(p, s_i) - \alpha w(p_{max}, s_i)] \\ = \quad & \int^{\Psi} [(V(\psi, s_i) - p)w'(p, s_i) \\ & -\frac{\rho}{2T(\psi, s_i)}w'^2(p, s_i) + \alpha w(p, s_i) - \alpha w(p_{max}, s_i)]g(\psi)d\psi, \end{aligned}$$

[36] When we formulate the bidders' optimization problem below, $p(\;)$ will be one of the state variables. In that context, $p_{max} = p(\Psi)$ is the price that corresponds to the maximum level of the sufficient statistic $\psi = \Psi$.

[37] In this paper, the prime ' is used to denote partial differentiation with respect to the first variable.

where:

$$\begin{aligned} V(\psi, s_i) &\equiv \mathrm{E}[\tilde{v}|\psi, s_i], \\ T(\psi, s_i) &\equiv 1/\mathrm{Var}[\tilde{v}|\psi, s_i]. \end{aligned}$$

While a variational approach is feasible, from this point on we model the ith bidder's optimization decision as a control problem with $w(\psi)$ and $p(\psi)$ as state variables, and $u_1(\psi) = w'(\psi)/p'(\psi)$ and $u_2(\psi) = p'(\psi)$ as control variables. The problem statement then becomes:

$$\begin{aligned} & \max_{w(\psi),\ p(\psi),\ u_1(\psi),\ u_2(\psi)} \int^{\Psi} [(V(\psi, s_i) - p(\psi))w'(p) - \frac{\rho T^{-1}(\psi, s_i)}{2} w'^2(p) \\ & \qquad + \alpha w(p) - \alpha w(p_{max})]g(\psi)d\psi \\ = & \int^{\Psi} [(V(\psi, s_i) - p(\psi))\frac{w'(\psi)}{p'(\psi)} - \frac{\rho T^{-1}(\psi, s_i)}{2} (\frac{w'(\psi)}{p'(\psi)})^2 \\ & \qquad + \alpha w(\psi) - \alpha w(\Psi)]g(\psi)d\psi \\ = & \int^{\Psi} [(V(\psi, s_i) - p(\psi))u_1(\psi) - \frac{\rho T^{-1}(\psi, s_i)}{2} u_1^2(\psi) \\ & \qquad + \alpha w(\psi)]g(\psi)d\psi - \alpha w(\Psi), \end{aligned}$$

subject to two state (transition) equations:

$$\begin{aligned} w'(\psi) &= u_1(\psi)u_2(\psi), \\ p'(\psi) &= u_2(\psi), \end{aligned}$$

the market clearing constraint:

$$1 - z = u_1(\psi) + \sum_{j \neq i} x(p(\psi), s_j), \tag{28}$$

and the following boundary conditions:

$$w(\Psi) > w_0, \quad p(\Psi) > p_0,$$

where w_0 and p_0 are constants.

We can "generate" the first-order conditions for the above control problem based on the following Lagrangian (the first two lines below are called the Hamiltonian):

$$\begin{aligned} L = & [(V(\psi, s_i) - p(\psi))u_1(\psi) - \frac{\rho T^{-1}(\psi, s_i)}{2} u_1^2(\psi) + \alpha w(\psi)]g(\psi) \\ & + \lambda_1(\psi)u_1(\psi)u_2(\psi) + \lambda_2(\psi)u_2(\psi) \\ & + \lambda_3(\psi)[u_1(\psi) + \sum_{j \neq i} x(p(\psi), s_j) + z - 1], \end{aligned} \tag{29}$$

where $\lambda_1(\psi)$, $\lambda_2(\psi)$, and $\lambda_3(\psi)$ are ψ-dependent Lagrangian multipliers.

The necessary conditions are the optimality equations:

$$\frac{\partial L}{\partial u_1} = 0, \qquad \frac{\partial L}{\partial u_2} = 0,$$

the adjoint equations:

$$\frac{\partial L}{\partial w} = -\lambda_1'(\psi), \qquad \frac{\partial L}{\partial p} = -\lambda_2'(\psi),$$

as well as the state equations and the market clearing constraint. In addition, any optimal solution to the problem must also satisfy the transversality condition:[38]

$$\lambda_1(\Psi) = -\alpha, \tag{30}$$
$$\lambda_2(\Psi) = 0. \tag{31}$$

The optimality equations and the adjoint equations are:

$$\begin{aligned}
[V(\psi, s_i) - p(\psi) - \rho T^{-1}(\psi, s_i)u_1(\psi)]g(\psi) & \\
+\lambda_1(\psi)u_2(\psi) + \lambda_3(\psi) &= 0, &(32)\\
\lambda_1(\psi)u_1(\psi) + \lambda_2(\psi) &= 0, &(33)\\
\alpha g(\psi) &= -\lambda_1'(\psi), &(34)\\
-u_1(\psi)g(\psi) + \sum_{j\neq i} x'(p(\psi), s_j)\lambda_3(\psi) &= -\lambda_2'(\psi). &(35)
\end{aligned}$$

Equation (34) can be integrated to yield: $\lambda_1(\psi) = -\alpha G(\psi)$, where the integration constant is fixed to be zero by Eq. (30). Combining the previous equation with Eq. (32), we have:

$$\lambda_3(\psi) = -[V(\psi, s_i) - p(\psi) - \rho T^{-1}(\psi, s_i)u_1(\psi)]g(\psi) + \alpha G(\psi)u_2(\psi).$$

Also, λ_2 can be solved from Eq. (33): $\lambda_2(\psi) = \alpha G(\psi)u_1(\psi)$.

Upon substituting the last three expressions for the Lagrangian multipliers into Eq. (35), we obtain the following:

$$\begin{aligned}
&-u_1(\psi)g(\psi) + \sum_{j\neq i} x'(p(\psi), s_j)\{-[V(\psi, s_i) - p(\psi)\\
&-\rho T^{-1}(\psi, s_i)u_1(\psi)]g(\psi) + \alpha G(\psi)u_2(\psi)\}\\
= \;&-\alpha[u_1(\psi)g(\psi) + G(\psi)u_1'(\psi)]. &(36)
\end{aligned}$$

Given our focus on symmetric strategy equilibria, we can make use of the following relations:

$$\begin{aligned}
u_1(\psi) &= \frac{w'(\psi)}{p'(\psi)} = w'(p) = y(p) = x(p, s_i),\\
u_1'(\psi) &= \frac{d}{d\psi}\left[\frac{w'(\psi)}{p'(\psi)}\right] = \frac{dx(p, s_i)}{d\psi} = x'(p, s_i)p'(\psi),
\end{aligned}$$

[38] See Kamien and Schwartz [13] for a treatment of dynamic optimization problems under various endpoint conditions.

to simplify Eq. (36), which becomes:[39]

$$
\begin{aligned}
& -x(p,s_i)g(\psi) - \sum_{j\neq i} x'(p,s_j)[V(\psi,s_i) - p(\psi) - \rho T^{-1}(\psi,s_i)x(p,s_i)]g(\psi) \\
& + \sum_{j\neq i} \alpha G(\psi)x'(p,s_j)p'(\psi) \\
= \; & -\alpha[x(p,s_i)g(\psi) + G(\psi)x'(p,s_i)p'(\psi)]. \qquad (37)
\end{aligned}
$$

In equilibrium, the market clearing condition Eq. (28) also implies the following (notice that ψ can always be scaled such that $\frac{\partial z}{\partial \psi} = 1$):

$$
1 = -x'(p,s_i)p'(\psi) - \sum_{j\neq i} x'(p,s_j)p'(\psi).
$$

Therefore, Eq. (36) in fact reduces to:

$$
\sum_{j\neq i} x'(p(\psi),s_j) = -\frac{(1-\alpha)x(p(\psi),s_i)g(\psi) + \alpha G(\psi)}{[V(\psi,s_i) - p(\psi) - \rho T^{-1}(\psi,s_i)x(p(\psi),s_i)]g(\psi)}. \qquad (38)
$$

□

B Proof of Propositions 3.3 and 3.6

In this appendix, we first present the seller's expected revenue for the general solutions of a uniform-price auction. We then compute the revenue for the unique (linear) solution of a discriminatory auction. Propositions 3.3 and 3.6 are based on these expressions.

Expected revenue

The seller's expected revenue in a uniform-price auction is:

$$
\begin{aligned}
R_{\alpha=0} &= \mathrm{E}[\tilde{p}] \\
&= \int_0^1 \{\bar{v}[1 - (\frac{1-z}{Nx_0})^{N-1}] \\
&\quad - \frac{(1-z)}{N}[1 - (\frac{1-z}{Nx_0})^{N-2}]\frac{(N-1)\rho\tau_v^{-1}}{N-2}\}g(z)dz. \qquad (39)
\end{aligned}
$$

This uses Eq. (9) and the fact that $x = (1-z)/N$ in symmetric equilibria.

Next we compute the seller's expected revenue in a discriminatory auction. In fact we derive a revenue expression that is valid for any linear strategy equilibrium in an auction with price discrimination parameter α. Suppose the bidding strategies are of the form:

[39] In this step, we used the transition equation $u_2(\psi) = p'(\psi)$.

$$p(x) = \frac{\mu - x}{\gamma}, \tag{40}$$

it is straightforward that expected payment by the N competitive bidders is:

$$\frac{N\mu(1 - \mathrm{E}[\tilde{z}]) - (1 - \alpha/2)\mathrm{E}[(1 - \tilde{z})^2]}{N\gamma},$$

and the noncompetitive bidders' expected payment to the seller is:

$$\frac{N\mu\mathrm{E}[\tilde{z}] - (1 - \alpha/2)\mathrm{E}[(1 - \tilde{z})\tilde{z}]}{N\gamma}.$$

Thus, the seller's expected revenue is the sum of these two expressions:

$$R_\alpha = \frac{N\mu - (1 - \alpha/2)(1 - \mathrm{E}[\tilde{z}])}{N\gamma}. \tag{41}$$

Using Eqs. (14) and (41), we have the following revenue for a discriminatory auction:

$$R_{\alpha=1} = \overline{v} - \frac{\rho\tau_v^{-1}\theta}{N(\theta + 1) - 1}[1 + \frac{N - 1}{2N(\theta + 1)}]. \tag{42}$$

On Proposition 3.3

For risk neutral bidders, we can set $\rho = 0$ in Eqs. (39) and (42). Thus:

$$R_{\alpha=1} \quad = \quad \overline{v} \;\geq\; R_{\alpha=0} = \int_0^1 \overline{v}[1 - (\frac{1 - z}{Nx_0})^{N-1}]g(z)dz,$$

where the equality holds for $x_0 = \infty$.

On Proposition 3.6

That $R_{\alpha=0}$ is increasing in x_0 can be directly seen from Eq. (39). The revenue comparison result for the risk averse case is based on numerical evaluations of the analytic expressions for the seller's expected revenue.

On certainty equivalent profit

Under symmetric bidder information, the certainty equivalent profit for competitive bidder i is:

$$\mathrm{CE}_i \quad = \quad \mathrm{E}[(\overline{v} - \tilde{p})\tilde{x}_i - \frac{\rho\tau_v^{-1}}{2}\tilde{x}_i^2 - \frac{\alpha}{2\gamma}\tilde{x}_i^2].$$

Using Eq. (9) and noting that $x = (1 - z)/N$ in symmetric equilibria, we can express the certainty equivalent profit from all N competitive bidders in a uniform-price auction as follows:

$$\begin{aligned}\mathrm{CE}_{\alpha=0} &= \int_0^1 \{\overline{v}(\frac{1-z}{Nx_0})^{N-1} + \frac{(1-z)}{N}[1-(\frac{1-z}{Nx_0})^{N-2}]\frac{(N-1)\rho\tau_v^{-1}}{N-2}\} \\ &\quad (1-z)g(z)dz - \frac{\rho\tau_v^{-1}}{2N}\int_0^1 (1-z)^2 g(z)dz. \qquad (43)\end{aligned}$$

With linear solutions of the form Eq. (40), the sum of certainty equivalent profit from all N competitive bidders is:

$$\mathrm{CE}_\alpha = (\overline{v} - \frac{\mu}{\gamma})\mathrm{E}[(1-\tilde{z})] + \frac{1-(\alpha+\gamma\rho\tau_v^{-1})/2}{N\gamma}\mathrm{E}[(1-\tilde{z})^2].$$

Using the above expression and Eq. (14), we have the following certainty equivalent profit for the discriminatory auction:

$$\mathrm{CE}_{\alpha=1} = \frac{\rho\tau_v^{-1}\theta^2}{(1+2\theta)[N(\theta+1)-1]}. \qquad (44)$$

Numerical computation based on Eqs. (43) and (44) confirms that $\mathrm{CE}_{\alpha=0} > \mathrm{CE}_{\alpha=1}$. □

C Proof of Corollary 4.1

The first statement is proved by showing that the converse is false. Suppose it is an equilibrium for the risk averse competitive bidders to submit flat bid schedules of the form $p = p(s_i)$, then the left-hand-side of Eq. (17) is (negative) infinity. Focus on the set of parameter values such that $(G/g)(\psi)$ is strictly positive, then the numerator of the right-hand-side of Eq. (17) is nonzero. Therefore, the denominator of the right-hand-side of Eq. (17) must be zero. That is:

$$p = V - \rho T^{-1}x.$$

Notice that V is a function of (ψ, s_i). Given our assumption that $\tilde{\psi}$ is a sufficient statistic for $\tilde{z}$ and $\tilde{s}_j$ in determining bidder i's residual supply curve, (ψ, s_i) is informationally equivalent to (p, s_i). Therefore, we can write the solution as:

$$p = V(p, s_i) - \rho T^{-1}(p, s_i)x.$$

This contradicts the conjecture of an equilibrium in which price $p = p(s_i)$ is independent of quantity x, except in the uninteresting case of a riskless asset.

The second statement of the corollary can be seen by examining Eq. (15). If there is a solution to the equation, $p = V(p, s_i)$, it represents the equilibrium of a first-price (unit) auction. For the case of a completely discriminatory auction this statement is, therefore, equivalent to Back and Zender's [2] Theorem 2. □

D Proof of Proposition 4.5

We sketch the proof in several steps. First we find a sufficient statistic for $\tilde{z}$ and $\tilde{s}_j$ for the purpose of determining bidder i's residual supply curve. We then apply Proposition 4.2 and solve for the symmetric, linear strategy equilibrium. In the next step, we verify the existence and uniqueness of the identified equilibrium. Lastly, we establish a numerical bound for a parameter (ϕ) that appears in the characterization of the equilibrium strategy.

Step 1. If $N-1$ competitive bidders (the bidder j's) submit symmetric, linear strategies of the form $\mu + \beta s_j - \gamma p$ and bidder i submits the bid schedule x_i, the market clearing condition is the following:

$$\begin{aligned} 1+f &= f+\tilde{z}+(N-1)\mu+\beta\sum_{j\neq i}(\tilde{v}+\tilde{\epsilon}_j)-(N-1)\gamma\tilde{p}+x_i \\ &= f+\tilde{z}+(N-1)\mu+(N-1)\beta\tilde{v}+\beta\sum_{j\neq i}\tilde{\epsilon}_j-(N-1)\gamma\tilde{p}+x_i\,. \end{aligned}$$

From the above equation, it is easy to show that $(\tilde{p},\tilde{s}_i)$ is informationally equivalent to $(\tilde{\psi},\tilde{s}_i)$ with:

$$\begin{aligned} \tilde{\psi} &= \frac{(N-1)\gamma\tilde{p}-(N-1)\mu+1-x_i}{(N-1)\beta} \\ &= \tilde{v}+\frac{1}{N-1}\sum_{j\neq i}\tilde{\epsilon}_j+\frac{1}{(N-1)\beta}\tilde{z} \;\equiv\; \tilde{v}+\tilde{\kappa}_i\,, \end{aligned}$$

where $\tilde{\kappa}_i$ is independent of $\tilde{\epsilon}_i$.

Using the properties of normally distributed random variables, we have:

$$\tau \;\equiv\; \mathrm{Var}^{-1}[\tilde{v}|\tilde{\psi},\tilde{s}_i] \;=\; \mathrm{Var}^{-1}[\tilde{v}|\tilde{v}+\tilde{\kappa}_i,\tilde{v}+\tilde{\epsilon}_i] \;=\tau_v+\tau_\kappa+\tau_e,$$

where

$$\tau_\kappa \;\equiv\; \mathrm{Var}^{-1}[\tilde{\kappa}_i] \;=\; \left(\frac{\tau_e^{-1}}{N-1}+\frac{\tau_z^{-1}}{(N-1)^2\beta^2}\right)^{-1} \;=\; (N-1)\tau_e\phi,$$

and

$$\phi=\frac{(N-1)\beta^2}{(N-1)\beta^2+\tau_e\tau_z^{-1}} \tag{45}$$

is a measure of *price efficiency*.

Similarly, we calculate the conditional expectation of the asset value as:

$$\begin{aligned} \mathrm{E}[\tilde{v}|\tilde{\psi},\tilde{s}_i] &= \mathrm{E}[\tilde{v}|\tilde{v}+\tilde{\kappa}_i,\tilde{v}+\tilde{\epsilon}_i] \;=\; \overline{v}+\frac{\tau_\kappa}{\tau}(\tilde{v}+\tilde{\kappa}_i-\overline{v})+\frac{\tau_e}{\tau}(\tilde{v}+\tilde{\epsilon}_i-\overline{v}) \\ &= \frac{\tau_v}{\tau}\overline{v}+\frac{\phi\tau_e[(N-1)\gamma\tilde{p}-(N-1)\mu+1-x_i]}{\beta\tau}+\frac{\tau_e}{\tau}\tilde{s}_i\,. \end{aligned}$$

To sum up the preceding discussion:

$$\mathrm{Var}^{-1}[\tilde{v}|\tilde{\psi}, \tilde{s}_i] \equiv \tau = \tau_v + \tau_e + (N-1)\phi\tau_e, \tag{46}$$

$$\begin{aligned} \mathrm{E}[\tilde{v}|\tilde{\psi}, \tilde{s}_i] &= \frac{\beta\tau_v\overline{v} + \phi\tau_e[1-(N-1)\mu - x_i]}{\beta\tau} \\ &\quad + \frac{\tau_e}{\tau}\tilde{s}_i + \frac{\phi\tau_e(N-1)\gamma\tilde{p}}{\beta\tau}. \end{aligned} \tag{47}$$

Step 2. We postulate the following linear bid schedule for all bidders $j \neq i$:

$$\tilde{x}_j = \mu + \beta\tilde{s}_j - \gamma\tilde{p},$$

where β is the sensitivity of bidder j's bids to his private signal, and γ the slope of his bid schedule.

Directly applying Eq. (17) of Proposition 4.2, we have the following relation:

$$-(N-1)\gamma = -\frac{x_i}{\mathrm{E}[\tilde{v}|\tilde{\psi}, \tilde{s}_i] - p - \rho\mathrm{Var}^{-1}[\tilde{v}|\tilde{\psi}, \tilde{s}_i]x_i},$$

which can be written as:

$$x_i = \frac{\mathrm{E}[\tilde{v}|p, s_i] - p}{\lambda + \rho\tau^{-1}},$$

a linear bid schedule, where

$$\lambda = \frac{1}{(N-1)\gamma}. \tag{48}$$

In a symmetric strategy equilibrium, i's bidding strategy is:

$$x_i = \mu + \beta s_i - \gamma p.$$

comparing coefficients by making use of Eqs. (46) and (47), we obtain:

$$\beta = \frac{\tau_e}{\lambda\tau + \rho + \phi\tau_e/\beta}, \tag{49}$$

$$\gamma = \frac{\beta\tau - \phi\tau_e(N-1)\gamma}{\beta(\lambda\tau + \rho + \phi\tau_e/\beta)}, \tag{50}$$

$$\mu = \frac{\beta\tau_v\overline{v} + \phi\tau_e[1-(N-1)\mu]}{\beta(\lambda\tau + \rho + \phi\tau_e/\beta)}. \tag{51}$$

Using Eq. (49) and the definition of τ, Eq. (50) can be written as:

$$\gamma = \frac{\beta\tau\tau_e^{-1}}{1+(N-1)\phi} = \beta\left(1 + \frac{\tau_v\tau_e^{-1}}{1+(N-1)\phi}\right). \tag{52}$$

Using Eqs. (48) and (52), we can rewrite Eq. (49) as:

$$1 - \phi = \rho\tau_e^{-1}\beta + \phi + \frac{1}{N-1}, \tag{53}$$

which is equivalent to the following cubic equation (notice Eq. (45)):

$$\rho\tau_e^{-1}\beta^3 + \frac{N}{N-1}\beta^2 + \frac{\rho\tau_z^{-1}}{N-1}\beta - \frac{(N-2)\tau_e\tau_z^{-1}}{(N-1)^2} = 0. \tag{54}$$

Finally, μ is solved from combining Eqs. (49), (51) and (52):

$$\mu = \bar{v}(\gamma - \beta) + \frac{\phi}{1+(N-1)\phi}. \tag{55}$$

We can sum up the solution to the competitive bidders' optimization problem as follows. First, β as a function of the exogenous parameters of the model is solved from Eq. (54). Then, Eqs. (45), (52) and (55) provide the rest of the characterization of the optimal symmetric, linear strategy.

Step 3. The existence and uniqueness of a positive β (and therefore of the other variables) can be established based on the fact that the cubic equation $y^3 + a_1 y^2 + a_2 y + a_3 = 0$ has one and only one positive root if $a_1 > 0$ and $a_3 < 0$. Since $N > 2$, the corresponding coefficients in Eq. (54) indeed have the appropriate signs.

Because both β and γ are strictly positive, the second-order condition of the bidders' optimization problem is always satisfied.

Step 4. From its definition, ϕ must be a number between 0 and 1. In this part, we show that a tighter upper bound exists for ϕ and its monotone transform:

$$I = \frac{1+(N-1)\phi}{1-\phi}. \tag{56}$$

Denote $\xi \equiv [1+(N-1)\phi]/N = \beta\tau/N\gamma\tau_e$, we can rewrite Eqs. (49) and (50) as:

$$\begin{aligned} \rho\beta\tau_e^{-1} &= 1 - \phi - \frac{N\xi}{N-1}, \\ 1-\xi &= \frac{(1-\phi)(N-1)}{N}. \end{aligned} \tag{57}$$

Therefore,

$$\frac{\rho\beta\tau_e^{-1}}{1-\phi} = \frac{1-2\xi}{1-\xi}.$$

Since the left-hand-side of the above equation is nonnegative and since ξ is no larger than 1 (from its definition), we have $\xi \leq 1/2$. Consequently, $\phi < (N/2-1)/(N-1)$ and $I < N-1$. Notice that ϕ is always less than one half. □

E Proof of Corollary 4.4

The equilibrium stop-out price can be solved from the market clearing condition:

$$\tilde{p} = \frac{N\mu - 1}{N\gamma} + \frac{\tilde{z} + \beta\sum_{j=1}^{N}\tilde{s}_j}{N\gamma}.$$

In addition, bidder i's bid schedule can be written as:

$$\tilde{x}_i = \frac{1}{N} + \beta\tilde{s}_i - \frac{\tilde{z} + \beta\sum_{j=1}^{N}\tilde{s}_j}{N}.$$

Using these expressions, it is straightforward to calculate:

$$\begin{aligned}
\mathrm{E}[\tilde{p}] &= \frac{N\bar{v}\beta + N\mu - 1}{N\gamma} = \bar{v} - \frac{1-\phi}{N\gamma[1+(N-1)\phi]} = \bar{v} - \frac{1}{N\gamma I},\\
\mathrm{E}[\tilde{x}_i] &= \frac{1}{N},\\
\mathrm{Var}[\tilde{p}] &= \frac{\tau_z^{-1} + N\beta^2\tau_e^{-1} + N^2\beta^2\tau_v^{-1}}{N^2\gamma^2},\\
\mathrm{Var}[\tilde{x}] &= \frac{\tau_z^{-1} + N(N-1)\beta^2\tau_e^{-1}}{N^2} = \frac{I\tau_z^{-1}}{N^2}.
\end{aligned}$$

In addition, we have:

$$\mathrm{E}[\tilde{x}_i\tilde{s}_i] = \frac{\bar{v} + (N-1)\beta\tau_e^{-1}}{N}.$$

Taking advantage of the symmetry of the problem, we can compute the expected payment received by the seller from the competitive bidders (denoted $\square$) as follows:

$$\square \equiv \sum_{i=1}^{N}\mathrm{E}[\tilde{x}_i\tilde{p}(\tilde{x}_i)] = N\,\mathrm{E}[\tilde{x}_i\tilde{p}(\tilde{x}_i)] = N\,\mathrm{E}[\tilde{x}_i\frac{\mu + \beta\tilde{s}_i - \tilde{x}_i}{\gamma}].$$

Using properties derived in Appendix D:

$$\begin{aligned}
\square &= \frac{N\bar{v}(\gamma-\beta) + (1-1/I)}{N\gamma} + \frac{N\beta(\bar{v} + (N-1)\beta\tau_e^{-1})}{N\gamma}\\
&\quad - \frac{N(1+\tau_z^{-1} + N(N-1)\beta^2\tau_e^{-1})}{N^2\gamma}\\
&= \bar{v} - \frac{I^{-1} + \tau_z^{-1}}{N\gamma}.
\end{aligned}$$

The total expected bidding profit for the N competitive bidders is:

$$N\pi \equiv \mathrm{E}[\tilde{v}\sum_{i=1}^{N}\tilde{x}_i] - \square = \mathrm{E}[\tilde{v}(1-\tilde{z})] - \square = \frac{I^{-1} + \tau_z^{-1}}{N\gamma}.$$

The expected bidding profit for the noncompetitive bidders is:

$$\begin{aligned}
\pi_{nc} \equiv \mathrm{E}[(f+\tilde{z})(\tilde{v}-\tilde{p})] &= f\bar{v} - f(\bar{v} - \frac{1}{N\gamma I}) - \frac{\tau_z^{-1}}{N\gamma}\\
&= \frac{f\,I^{-1} - \tau_z^{-1}}{N\gamma} \equiv f\bar{v} - R_{nc},
\end{aligned}$$

where R_{nc} denotes the expected payment received by the seller from the non-competitive bidders.

The seller's total expected revenue is therefore:

$$R \quad \equiv \quad \square + R_{nc} \; = \; (1+f)\overline{v} - N\pi - \pi_{nc} \; = \; (1+f)\overline{v} - \frac{(1+f)I^{-1}}{N\gamma}.$$

We observe that, *ceteris paribus*, R is increasing in both I and γ. □

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