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Auctions of divisible goods with endogenous supply

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Abstract

Uniform-price auctions are studied in which the seller may cancel part of the supply after observing the bids. This feature eliminates many of the 'collusive seeming' equilibria of the auction. In equilibrium the seller always sells the full quantity.

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1. Introduction

An important but largely unexplored issue in multi-unit auctions concerns the impact of the seller retaining an option to adjust the quantity sold after observing the bids. This is a feature of Treasury auctions in some countries.[1] Back and Zender (1993), extending Wilson (1979), show that when the quantity is fixed and known, there exist equilibria of a uniform-price auction that are arbitrarily bad for the seller. In these equilibria, bidders ignore their own information concerning value and the outcome is characterized by symmetric allocations at prices (possibly far) below the value of the good. There are equilibria with similarly bad outcomes, from the seller's point of view, when supply is uncertain — see Back and Zender (1993) and Wang and Zender (2001).[2]

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[1]Umlauf (1993) and Heller and Lengwiler (1998) report that the Mexican and Swiss (respectively) Treasuries retain this right.

[2]However, Swinkels (2001) shows that if there is sufficient noise in the auction so that the equilibria satisfy a condition he calls Asymptotic Environmental Similarity, then bids converge to marginal values in uniform-price private-value auctions as the number of bidders goes to infinity.

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Such outcomes can be supported in a uniform-price auction because bidders are concerned with a single point on their demand curves, the one corresponding to the stop-out price. The rest of a bidder's demand curve is left unrestricted and can be used to inhibit competition from other bidders.

In a discriminatory auction, this is not the case. Each price bid by any bidder (above the stop-out price) will be paid by that bidder. This means each bidder has a direct interest in his entire demand curve. The use of a discriminatory auction effectively acts as a restriction on the set of strategies that may be played in equilibria. The right of the seller to restrict the supply ex post in a uniform-price auction acts to restrict the equilibrium strategies of the bidders in a similar way. Each bidder is concerned not only with the stop-out price that would occur given no supply restriction but also with any point on his demand curve that might be hit due to an ex post supply restriction.

The seller finds it advantageous to reduce supply only when the aggregate demand curve is steep. Equilibria supported by steep demand curves are therefore vulnerable to a supply restriction. This limits how steep each individual demand curve may profitably be and so reduces the extent to which any bidder may inhibit competition from his rivals.

Only a few studies have examined auctions with an endogenous quantity. Hansen (1988) examines a model in which *sellers* compete in a unit auction for the right to sell to a market with a downward sloping aggregate demand curve. Lengwiler (1998) considers a divisible good auction, with two possible bid levels, in which a seller, who produces the good at a constant marginal cost, may vary the quantity sold (up or down) after the bids have been submitted. Ausubel and Cramton (1999) examine the impact of restricting supply or reserve pricing on the revenue derived in a Vickrey auction. McAdams (2000) independently obtains a result similar to ours.

2. Model and results

We use the model of Back and Zender (1993), except that we assume common knowledge by the bidders of the asset value v. There are $n > 1$ bidders and a single seller. The good is assumed to be perfectly divisible and a maximum quantity $\bar{Q}$ is available to be sold.

For simplicity, we assume the seller sets a reserve price of zero. Nothing in the paper would be materially changed if there were a positive reserve price less than v. Each bidder i submits a demand schedule. A demand schedule is a nonincreasing left-continuous function $q:[0, \infty) \rightarrow [0, \bar{Q}]$. Denote the demand schedule of bidder i by q_i. After observing the demand schedules $q_1, \ldots, q_n$, the seller chooses the actual quantity $Q \leq \bar{Q}$.

In the auction, all bids at or above the stop-out price are accepted. The stop-out price is the maximum price at which demand equals or exceeds the supply Q or the reserve price if there is excess supply at all prices. Formally, the stop-out price is

$$\pi(Q) = \max\left\{ p \middle| \sum_{i=1}^{n} q_i(p) \geq Q \right\} \tag{1}$$

with the convention that the maximum of the empty set is zero.

If there is a discontinuity (flat) in the aggregate demand curve at the stop-out price, then it may be necessary to ration the demands. We assume there is pro rata rationing of marginal bids.[3] This is defined as follows. The flat in an individual's demand curve is $\Delta q_i(p) = q_i(p) - \lim_{p' \downarrow p} q_i(p')$, and the flat in the aggregate demand curve is $\Sigma_{i=1}^{n} \Delta q_i(p)$. The fraction of the flat in the aggregate demand curve that cannot be filled is

$$\lambda(Q) = \frac{\sum_{i=1}^{n} q_i(\pi(Q)) - Q}{\sum_{i=1}^{n} \Delta q_i(\pi(Q))}, \tag{2}$$

so the quantity received by bidder i is

$$\theta_i(Q) \equiv q_i(\pi(Q)) - \lambda(Q)\Delta q_i(\pi(Q)). \tag{3}$$

In a uniform-price auction, each bidder pays the stop-out price, so his payment is $\pi(Q)\theta_i(Q)$. We assume that the seller gets no utility from the asset being sold (in particular, he cannot sell any reserved quantity in another auction or in an after-market for the asset). This is an obviously unrealistic but simplifying assumption. In consequence, we assume that the seller's objective is to maximize revenue in the auction, so, after observing the bids, he chooses $Q \le \bar{Q}$ to maximize $\Sigma_{i=1}^{n} \pi(Q)\theta_i(Q)$.

The following is the result from Back and Zender (1993) that shows the uniform-price auction has equilibria that are arbitrarily bad for the seller when the seller cannot restrict the supply after observing the bids. Throughout this note, 'equilibrium' will mean a pure-strategy subgame-perfect equilibrium that is symmetric with respect to the bidders.

Theorem 1. *Assume the seller is constrained to choose $Q = \bar{Q}$. For each $p^* \le v$, there is an equilibrium in which the stop-out price is p^*. An equilibrium demand curve is*

$$q_i(p) = \begin{cases} 0 & \text{if} \quad p > p^\dagger \\ \dfrac{(p^\dagger - p)}{n(p^\dagger - p) + p - p^*}\bar{Q} & \text{if} \quad p^* < p \le p^\dagger \\ \bar{Q}/(n-1) & \text{if} \quad 0 \le p \le p^*, \end{cases} \tag{4}$$

where $p^\dagger = (n-1)v/n + p^*/n$.

In contrast, when the seller may restrict supply, we obtain the following.

Theorem 2. *Assume the seller can choose $Q \le \bar{Q}$ after observing the bids. In any equilibrium, the seller chooses $Q = \bar{Q}$ and the stop-out price is at least $(n-1)v/n$.*

Proof. Denote the equilibrium demand curve of each bidder by q^*. Let π denote the function (1)

[3] This is the procedure used by the US Treasury. Kremer (1998) and Nyborg (1999) show that rationing of all bids at or above the stop-out price leads to better outcomes for the seller in a uniform-price auction.

when each bidder plays q^*. Let Q^* denote the equilibrium quantity chosen by the seller and let $p^* \equiv \pi(Q^*)$ denote the equilibrium stop-out price.

By the definition of π, the aggregate demand $nq^*(\pi(Q))$ equals or exceeds the quantity Q whenever $\pi(Q) > 0$; furthermore, the full quantity $\bar{Q}$ is allocated to the bidders when $\pi(\bar{Q}) > 0$. The seller's revenue is therefore $\pi(Q)Q$ for any $Q \leq \bar{Q}$ and by revenue maximization we have $p^*Q^* \geq \pi(Q)Q$. This implies

$$(\forall Q \leq \bar{Q}) \quad \frac{p^*}{Q} \geq \frac{\pi(Q) - p^*}{Q^* - Q}. \tag{5}$$

The right-hand side of this is the absolute value of the slope of the aggregate demand curve. We will derive a lower bound on this slope by considering the behavior of the bidders, and the two bounds taken together will deliver the result about the equilibrium stop-out price.

We will consider deviations from the equilibrium strategy by any bidder i. The deviations will be of the form

$$q_i(p) = \begin{cases} q^*(p) & \text{if} \quad p > \hat{p}, \\ \bar{Q} & \text{if} \quad p \leq \hat{p} \end{cases} \tag{6}$$

for some $\hat{p}$. We will denote the function (1) when bidder i plays this deviation by $\hat{\pi}$. Note that $\hat{\pi}(Q) = \pi(Q)$ if $\pi(Q) > \hat{p}$ and $\hat{\pi}(Q) = \hat{p}$ otherwise. This implies $\hat{\pi}(Q) = \max\{\pi(Q), \hat{p}\}$. It follows that $\sup_Q \hat{\pi}(Q)Q = \max\{p^*Q^*, \hat{p}\bar{Q}\}$. Hence, faced with this deviation, the seller will choose the quantity $\bar{Q}$ and the stop-out price will be $\hat{p}$ if $\hat{p}\bar{Q} > p^*Q^*$. In particular, the seller will choose the quantity $\bar{Q}$ and the stop-out price will be $\hat{p}$ if $\hat{p} > p^*$.

First, we will show that $Q^* = \bar{Q}$. Suppose to the contrary that $Q^* < \bar{Q}$. Then $nq^*(p^*) = Q^* < \bar{Q}$. Consider a deviation by i of the type described above where $\hat{p} = p^* - \varepsilon$. If $\varepsilon > 0$ is sufficiently small, the seller will choose the quantity $\bar{Q}$, the stop-out price will be $\hat{p}$, and bidder i's allocation will approximate $q^*(p^*) + \bar{Q} - Q^*$. The larger allocation at price $\hat{p} < p^* \leq v$ would increase bidder i's profits, so we conclude that we cannot have $Q^* < \bar{Q}$.

For the remainder of the proof, we can assume $p^* < v$. We will need to know that the aggregate demand $nq^*(p)$ converges to $\bar{Q}$ as $p \downarrow p^*$. Let $Q^\dagger = \sup\{nq^*(p) | p > p^*\}$. Consider a deviation by bidder i where $\hat{p} = p^* + \varepsilon$ for $\varepsilon > 0$ such that $p^* + \varepsilon < v$. The seller will choose quantity $Q = \bar{Q}$, the stop-out price will be $\hat{p}$, and if ε is sufficiently small, bidder i's allocation will approximate $Q^\dagger/n + \bar{Q} - Q^\dagger$. The larger allocation at only a slightly higher price would increase bidder i's profits if $Q^\dagger < \bar{Q}$, so we conclude that $Q^\dagger = \bar{Q}$, which means that $nq^*(p) \to \bar{Q}$ as $p \downarrow p^*$.

Now we are ready to derive the lower bound on the slope of the aggregate demand curve. Consider a deviation by bidder i with $\hat{p} > p^*$. This deviation results in bidder i receiving the quantity $\bar{Q} - (n-1)q^*(\hat{p})$ at price $\hat{p}$. This must be less profitable than his equilibrium strategy, so

$$(v - \hat{p})[\bar{Q} - (n-1)q^*(\hat{p})] \leq (v - p^*)\bar{Q}/n. \tag{7}$$

After some manipulation, if we set $Q = nq^*(\hat{p})$, then by definition $\pi(Q) = \hat{p}$. Making this substitution yields

$$\frac{\pi(Q) - p^*}{\bar{Q} - Q} \geq \frac{v - p^*}{\frac{n}{n-1}\bar{Q} - Q}, \tag{8}$$

which is the desired lower bound on the slope of the aggregate demand curve.

Combining (5) and (8) gives us

$$\frac{p^*}{Q} \geq \frac{v - p^*}{\frac{n}{n-1}\bar{Q} - Q}, \tag{9}$$

where $Q = nq^*(\hat{p})$ and $\hat{p} > p^*$. This implies

$$p^* \geq \frac{n-1}{n}\left(\frac{Q}{\bar{Q}}\right)v. \tag{10}$$

We have already seen that the aggregate demand $Q \equiv nq^*(\hat{p})$ converges to $\bar{Q}$ as $\hat{p} \downarrow p^*$. Taking this limit yields $p^* \geq (n-1)v/n$. $\square$

Despite the fact that the seller's right to restrict the supply of the good ex post is not used in equilibrium, it carries a very real benefit for the seller. It places a limit on the steepness of the aggregate demand curve that restricts the bidders' ability to inhibit competition from their rivals. Theorem 2 shows that there is a benefit to increasing the number of bidders in the uniform price auction with supply restrictions while there was no such benefit under the conditions of Theorem 1.

3. Conclusion

This analysis may help to reconcile seemingly contradictory findings regarding the use of uniform-price versus discriminatory auctions. For example, the theoretical analysis in Back and Zender (1993) and the empirical analysis of the US experience with uniform-price versus discriminatory auctions contained in Simon (1992) support the use of discriminatory auctions. Umlauf's (1993) analysis of the Mexican experience with these same pricing rules indicates that the uniform-price auction is revenue superior. Recall, however, that the Mexican Treasury retains the (little used) right to restrict the supply of bonds ex post.

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