Is the Parrondo Paradox a Paradox?

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What in Hell is the Parrondo Paradox?

- The ***Parrondo Paradox*** is that:

  “Given two games, each with a higher probability of winning than losing, it is possible to construct a winning strategy by playing the games alternately.”

  The Wikipedia sez:

  “It has been argued that Parrondo’s games are of little practical use…Work on finding connections to the stock market is now underway.”

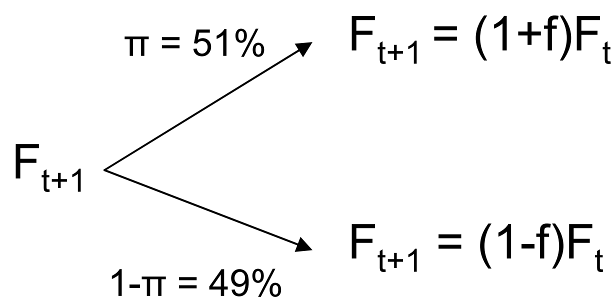
Objectives

- I will show that garden variety diversification strategies provide the mechanism for Parrondo Paradoxes in binomial models of gambling and investment.

- I will present experimental evidence to help determine whether or not people are surprised by the phenomenon.

A Textbook Investment Process

- Finance texts use a *binomial process to* model the IID evolution of a stock's return.
- EXAMPLE: I will use an isomorphic example from the gambling literature [Thorp (1984), MacLean, Ziemba, and Blazenko (1992)]
  - Gambler bets f% of funds each play, wins with probability $\pi = 51\%$ or loses with $1\text{-}\ \pi = 49\%$



Probabilistic Analysis

- Letting "w" denote the number of wins:

$$F_n = F_0[(1+f)^w(1-f)^{w-n}]$$

$$E[F_n] = E[F_1]^n = F_0[\pi(1+f)+(1-\pi)(1-f)]^n$$

- Suppose $\pi = 51\%$ and **f = 5%,** and n = 1000 plays. Then

$$E[F_n] = 1.001^{1000} F_0 = 2.72 F_0$$

So this looks like a good bet. But wait…

…The Distribution is Highly Skewed to the Right.

• As a result, Median[F_n] << E[F_n] . Because the Median of a monotone increasing function of w is that same function of the Median:

$$
\begin{aligned}
F_n &= F_0[(1+f)^w(1-f)^{n-w}] \\
&= F_0 e^{\log[(1+f)^w(1-f)^{n-w}]} \\
&= F_0 e^{w\log(1+f)+(n-w)\log(1-f)} \\
&= F_0 e^{[\frac{w}{n}\log(1+f)+\frac{n-w}{n}\log(1-f)]n}
\end{aligned}
$$

$$
Median[F_n] = F_0 e^{[\frac{n\pi}{n}\log(1+f)+\frac{n-n\pi}{n}\log(1-f)]n}
$$

$= 0.778\ F_0$

So Losing is More Likely Than Winning !!!

$ Left After 2,000 Plays	**Percentage of Simulations**
0 - 10	22.4%
10 - 50	25.3%
50 - 100	12.3%
100 - 200	9.9%
200 - 400	10.2%
400 - 800	7.3%
800 - 1600	4.9%
1600 - 3200	3.6%
3200 -	4.1%

Starting with $100

The Kelly Bet Will Not Do That!

$$Median[F_n] \quad = \quad F_0 e^{[\frac{n\pi}{n}\log(1+f)+\frac{n-n\pi}{n}\log(1-f)]n}$$

Kelly Bet f^* maximizes Median, i.e. maximizes numerator. [Kelly criterion - Wikipedia, the free encyclopedia]

$$f^* = \pi - (1-\pi) = 2\%$$

$$\text{Median}[F_n^*] > F_0$$

But, we will see that the Paradox can arise from overbetting, e.g. f = 5% as assumed herein.

Can the Median Loss Be Turned Into a Gain by Investing in Two of These Losers?

- I consider two ways of "playing" two of these "games" at once:

1. Split your initial stake in half. Put each half in a game. Let the two games run.
   - This models diversified "buy-and-hold" investing, because the two games' returns are independent (and hence uncorrelated) and the money is left in each for the long-run.

     Wikipedia Horizontal Diversification
2. Again, split your initial stake in half. Put each half in a game. But after each play of both games, reallocate your funds so that half is still available for betting on the next play in each game.
   - This models diversified "rebalanced" investing, because we always keep the same fractional "asset allocation" throughout (i.e.50-50) balance of the two "investments".

     Wikipedia Rebalancing Bonus

Diversified "Buy-and-Hold"

Now suppose the bettor plays two identical games alternately. He diversifies, placing half his funds, i.e. \$5000, in each game. He then alternates play between the two, and lets the funds "ride" in each game. His total fortune after n plays of both games (determined by $2n$ Bernoulli trials) is:

$$F_n(Diversified\ Ride) = \frac{1}{2}F_0[(1+f)^{w_1}(1-f)^{w_1-n}] + \frac{1}{2}F_0[(1+f)^{w_2}(1-f)^{w_2-n}] \tag{5}$$

Because $\pi = .51$ is close to half, the binomially distributed number of wins is approximately normally distributed, and using the last line in (1), we see that the sum of two independent log normal distributions may be used to approximate the distribution of (5). Unfortunately, that distribution does not have a closed form. So $10,000$ simulations of (5) were performed to estimate the median of (5), resulting in

$$Median[F_n(Diversified\ Ride)] \approx 1.2 * F_0 \tag{6}$$

Hence, we see that 0.778 F_0 is increased to 1.2 F_0 by diversification!
Parrondo's Paradox is produced by a diversified "buy-and-hold" policy.

$ Left After 2,000 Plays	Percentage of Simulations
0 - 10	7.3%
10 - 50	23.3%
50 - 100	14.6%
100 - 200	15.8%
200 - 400	14.0%
400 - 800	10.0%
800 -1600	7.0%
1600 - 3200	4.3%
3200 -	3.7%

Starting with $100: Buy and Hold Diversification

Rebalanced Diversification

- No matter which of the two games wins or loses after one play, split the total of the two equally between them. Then bet the same fraction f in each. Keep this up.
  - This is akin to "selling" some of the winning game, and using the funds to "buy" some of the losing game

Outcome Probability Analysis:

$$\frac{1.05F_t}{2} + \frac{1.05F_t}{2} = 1.05F_t \; with \; probability = \pi^2$$

$$\frac{1.05F_t}{2} + \frac{0.95F_t}{2} = 1.0F_t \; with \; probability = 2\pi(1-\pi)$$

$$\frac{0.95F_t}{2} + \frac{0.95F_t}{2} = 0.95F_t \; with \; probability = (1-\pi)^2$$

Note the reduction in variance caused by the middle term!!

Diversified Rebalancing Rules!

$$E[F_n(Rebalanced\ Diversification)] = F_0[\pi^2(1+f)+(1-\pi)^2(1-f)+2\pi(1-\pi)(1)]^n$$

$$= F_0(1.001)^{1000} = 2.717F_0$$

This is the same expected value as in the single game.

$$Median[F_n] \approx F_0 e^{[\pi^2 \log(1+f)+(1-\pi)^2 \log(1-f)+2\pi(1-\pi)\log(1)]}$$

$$= F_0 e^{[.0003748*1000]} = 1.45F_0$$

$1.45F_0$ >> $0.778F_0$ (the single game's median) and also greater than the $1.2F_0$ median that resulted from buy-and-hold diversification.

$ Left After 2,000 Plays	Percentage of Simulations
0 - 10	2.7%
10 - 50	16.5%
50 - 100	14.0%
100 - 200	16.6%
200 - 400	16.7%
400 - 800	13.7%
800 -1600	10.0%
1600 - 3200	5.8%
3200 -	4.0%

Starting with $100, Rebalanced Diversification

Median Vs. Mean: General Results

- When returns $1+R_t$ are IID, Ethier (*J. Applied Prob.,2004)* showed that a good approximation to the median for suitably large values of n is:

$$\text{Median}[F_n] \approx F_0 \, \text{EXP}\{E[\text{Log } 1+R]\, n + \text{Skew}[\text{Log } 1 + R]\}$$

- Because the first term is multiplied by n while the latter term isn't, the latter (i.e. skewness) term may often be ignored.

- So the large-n Median is determined mainly by E[log 1 + R], not E[1+R]…that only determines the (high but atypical) Mean.

Generalizations

- Use a 2nd Order Approximation of the Log Gross Return Log(1+R) per period:

$$E[Log(1+R)] \approx E[R] - \frac{Var[R]}{2}$$

- Diversification of either type lowers Var[R], thus raising E[Log(1+R)], and hence the median.

  - As we saw, this can be strong enough to produce a Parrondo Paradox.

•This is a useful argument when explaining the rationale for diversification.

Experimental Evidence

- Qualtrics Sample of n = 172 plain people
  - About equal split of men and women
  - Median age: 52 years old
  - Few experts, or readers of financial press
- We surveyed their willingness to take:
  - A single f = 5% bet
  - 2000 sequential 5% bets (letting it ride)
  - After seeing the distribution, we polled again
  - 2000 sequential *Diversified* bets
  - After seeing *that* distribution, we polled again

Some Survey Results

- We ran cumulative logit models for a j =1,…7 point scale of acceptance [“Not at All” to “Bring it On”] used for each question Q:

$$logit[P(Q \leq j)] = \alpha_j + \sum_i \beta_i X_i$$

- Aging is positively related to long-run sequential betting
- Experience positively related to buy-and-hold diversification
- Self-rated Expertise is negatively related to buy-and-hold diversification

Some More Survey Results

- 52% strongly against the single bet
- Only **38%** strongly against long-run sequential bets
  - BUT, **46%** strongly against AFTER seeing Distribution

- 40% strongly against long-run Buy and Hold Diversified
  - No difference AFTER seeing distribution

- ***48%*** strongly against long-run Rebalanced Diversified
  - BUT, ***33%*** strongly against AFTER seeing Distribution

Conclusion: <u>Some folks change their minds after seeing the distribution, a consequence of the Paradox</u>

CONCLUSIONS:

- Parrondo Paradoxes are not just esoteric mathematical toys. They may easily arise in the most elementary betting and investment situations.

- In those situations, the paradox arises from a misunderstanding of the (limited) role played by the expected value.
  - Cumulative returns are heavily skewed to the right, so the mathematical expected value is an atypical outcome.
  - The median outcome is (by definition) not atypical, yet it is also influenced by the variance, not just the expected value.
  - The variance can be dramatically reduced by diversification, thus raising the median, possibly from negative to positive!

- Survey results (including others not reported here) indicate that people – even experts – are surprised to see that positive expected returns can cumulate to negative median returns, and that diversification is powerful enough to reverse that.