A Simple Parrondo Paradox

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Abstract

The Parrondo Paradox is a counterintuitive probabilistic phenomenon in situations of repeated decision making with uncertainty. Simply stated, it is that given two games, each with a higher probability of losing than winning, it is possible to construct a winning strategy by playing the games alternately. The best-known and analyzed examples are somewhat involved, and may not be relevant to practical situations of repeated decision making, e.g. casino games and investment in securities markets. Here, we construct a very simply stated and analyzed example of the Parrondo Paradox, using a model of stylized Blackjack-type betting that is well-known in the gambling literature.

1 Introduction

The Cambridge Dictionary online defines a paradox to be “a situation or statement which seems impossible or is difficult to understand because it contains two opposite facts or characteristics”. Some well-known paradoxes are statistical phenomena. For example, Lindley’s Paradox of statistical inference occurs when a classical test of the null hypothesis strongly rejects it, while a Bayesian posterior odds ratio highly favors it. [10] Of course, the phenomenon can be understood; it is just difficult to do so without further analysis. Another example is Parrondo’s Paradox, posed as follows:

> But can two losing gambling games be set up such that, when they are played one after the other, they becoming winning? The answer is yes. [5]

Key, Klosek, and Abbott [7] demonstrate this in situations where play switches from one simple game to another whenever the total fortune is divisible by an integer determined in advance. They generalize this to incorporate alternative switching rules, including the possibility of switching plays on the result of a coin toss. Di Crescenzo [2] develops a Parrondo Paradox in the reliability of chained components, by exploiting the fact that the product of mixtures of survival functions stochastically dominates the product of the component survival functions. Berresford and Rockett [1] contains a nice exposition of the earlier Parrondo Paradox examples, while Ethier and Lee [4] devise a framework for generalizing and analyzing the earlier examples. Here, we demonstrate that a Parrondo Paradox may arise in a simple model of Blackjack-type betting, which has been discussed by Ed Thorp [11], the famous inventor of card-counting Blackjack strategies. The required switching rules are also in accord with reality – they are the two most common ways of implementing the oft-advocated principle of diversification.

2 "Blackjack"-type Betting

The popular stylized model of "Blackjack" betting, discussed by Thorp [11, chap.9], MacLean, et.al. [8, p. 1575-1577] and others is now described. The bettor chooses a constant fraction f of her table fortune to bet on each (Bernoulli) play, having a gross return per dollar invested equal to $1+f$ with probability $\pi > 1/2$, or $1-f$ with probability $1-\pi$. It is assumed that the bet is favorable, i.e. $\pi > 1/2$, so the bet has a positive "edge" $(2\pi - 1)$ and positive expected net return $(2\pi - 1)f$ per play. Hence by what is popularly referred to as "Arrow's Theorem", any risk-averse (i.e. expected strictly concave utility of return) bettor will want to bet *some* fraction of fortune $f > 0$.

For example, suppose a person starts with $F_0 = \$10,000$ and chooses to bet $f = 5\%$ on each play. So he bets $10,000f = \$500$ on the first play. If he wins, he will have $10,000(1.05) = 10,500$ available for the next play. He then bets $10,500f = 525$. If he wins again, he will have $10,500(1.05) = \$11,025$, but if he loses, he will have $10,500(0.95) = 9975$. These Bernoulli trials continue.

MacLean,et.al. (op.cit., p. 1576) report the well-known finding that successful "card counting" Blackjack strategy can result in a favorable edge, e.g. see Thorp (op.cit.), and that "an approximation that will provide us insight into the long-run behavior of a player's fortune is to simply assume that the game is a Bernoulli trial with a probability of success $\pi = .51$ and probability of loss $1 - \pi = .49$."

The bettor's fortune after n plays, denoted F_n, will be

$$\begin{aligned} F_n &= F_0[(1+f)^w(1-f)^{n-w}] \\ &= F_0 e^{\log[(1+f)^w(1-f)^{n-w}]} \end{aligned}$$

$$= F_0 e^{w \log(1+f)+(n-w)\log(1-f)}$$

$$= F_0 e^{[\frac{w}{n}\log(1+f)+\frac{n-w}{n}\log(1-f)]n} \tag{1}$$

where w is the binomially distributed number of winning bets. The bettor's fortune F_n is the product of n i.i.d. random variables, so its expected value after n plays is the compounded value of its expected value per play, i.e.

$$E[F_n] = F_0[\pi(1+f) + (1-\pi)(1-f)]^n. \tag{2}$$

With $\pi = 51\%$ and $f = 5\%$, $E[F_n] = 1.001^n F_0$, so after, say, $n = 1000$ repeated plays, the expected value of the fortune is $2.717F_0$. But this statistic is extremely misleading. The fortune F_{1000} is extremely positively skewed. Too see this, recall that the binomially distributed number of wins w is approximately normal when π is close to 0.5 and n is suitably large. The last line of (1) then shows that F_n is approximately lognormal, and thus positively skewed. As such, its expected value is atypically high, much higher than the lower median. In fact, Ethier [3] shows that for suitably large n in many i.i.d. processes (including this one), a reasonable approximation to the median wealth is made by using the expected log gross return per play as an asymptotic exponential growth factor for the median. The approximation is

$$Median[F_n] \approx F_0 e^{[\pi \log(1+f)+(1-\pi)\log(1-f)]n}$$

$$= F_0 e^{[-.00025]1000} = 0.778F_0. \tag{3}$$

that is, the approximate median outcome is a *loss* of around 22% of his initial stake. Because the median outcome is a loss of fortune, there is a higher probability of losing income in the repeated game than in winning income. This is quite disturbing, and illustrates both the risk of overly aggressive betting in a favorable game and the spectacular failure of the expected value (172% gain) as a measure of central tendency in a highly skewed distribution of outcomes.

An alternative confirmation of this is enabled by simulating values of F_n on a computer spreadsheet. Its random number generator may be used to simulate the 51% biased coin toss. Each time a "head" is tossed, multiply F_{n-1} by 1.05. If a "tail" is tossed, multiply F_{n-1} by 0.95. Starting with $F_0 = 10,000$ and doing this for 1000 simulated tosses results in a simulated value for F_{1000}. Figure 1 depicts a histogram from 234 simulated values of F_{1000}, illustrating the positive skewness, and showing that the resulting simulated median is far lower than the population expected value near 27120. Of course, there is simulation error in the simulated value for the median, but the cumulated histogram shows that the simulated median is close to the value calculated in (3).

Insert Figure 1 Near Here

Even without the results of this simulation, readers should not worry about the accuracy of the approximation (3). To see why, note that the median number of wins is (the closest integer to) $n\pi = 510$, and that (1) is a monotone increasing transformation g of the number of wins w. Because $Median[g(w)] = g[Median[w]]$ for any g that is monotone increasing, (1) implies that

$$\begin{aligned} Median[F_n] &= F_0 e^{[\frac{n\pi}{n}\log(1+f)+\frac{n-n\pi}{n}\log(1-f)]n} \\ &= F_0 e^{[-.00025]1000} = 0.778 F_0 \end{aligned} \tag{4}$$

showing that the approximation (3) for i.i.d. plays is exact in our biased coin-toss special case. In other i.i.d. cases, Ethier (op.cit.) shows that an even more accurate approximation incorporates the degree of skewness in the log gross return per play, but this refinement was obviously not needed here.

3 Diversification Mechanisms Producing a Parrondo Paradox

Now suppose the bettor plays two identical games alternately. He diversifies, placing half his funds, i.e. \$5000, in each game. He then alternates play between the two, and lets the funds "ride" in each game. His total fortune after n plays of both games (determined by $2n$ Bernoulli trials) is:

$$(5) \qquad F_n(Diversified\ Ride) = \frac{1}{2}F_0[(1+f)^{w_1}(1-f)^{w_1-n}] + \frac{1}{2}F_0[(1+f)^{w_2}(1-f)^{w_2-n}]$$

Because $\pi = .51$ is close to half, the binomially distributed number of wins is approximately normally distributed, and using the last line in (1), we see that the sum of two independent log normal distributions may be used to approximate the distribution of (5). Unfortunately, that distribution does not have a closed form. So the aforementioned spreadsheet simulation was modified to simulate values of (5). Just split $F_0 = 10,000$ in half and perform the simulation as above, to produce a simulated value of the first term in (5). Do the same thing to produce a simulated value of the second term of (5), and then just add the two results to produce a simulated value of (5). To reduce simulation error, I did this $10,000$ times, and found the simulated median

$$(6) \qquad Median[F_n(Diversified\ Ride)] \approx 1.2 * F_0.$$

Summarizing, we found that repeated play of the single game resulted in a median loss of around 22% of the initial stake, while merely splitting the initial stake between two such identical games resulted in a median gain of around 20%! The outcome after repeated play of either game is a higher probability of losing money than winning it. But splitting the initial stake between two such games, and playing them alternately in the same way that the single game was played, (that is, betting $f = 5\%$ of the income at risk in each game) results in a higher probability of winning income than losing it. This is a Parrondo Paradox.

The probabilistic mechanism underlying this result is simple. When only one game is played, the possible outcomes of the first play are just a gain of 5% with 51% probability or a loss of 5% with 49% probability. But when the money is split between the two games, there are three possible outcomes of the first play of both: a gain of 5% from $(5000 * 1.05 + 5000 * 1.05)$ with $51\%^2 = 26.01\%$ probability , a loss of 5% with probability $49\%^2 = 24.01\%$, and breaking even (from $5000 * 1.05 + 5000 * 0.95 = 10,000$) with the complementary probability 49.98% (from $2 * .51 * .49$). The nearly even odds of breaking even lowers the volatility of the first play, while leaving the expected value of the first play unchanged. This mechanism works to raise the median of (5).

But wait: it is possible to do even better. Suppose the bettor splits his initial stake equally between two identical and independently run "Blackjack"games, but now commits to maintaining the 50-50 split of funds over the course of play. That is, he again starts with $F_0 = \$10000$, putting \$5000 to work in one game and the other \$5000 to work in the other game. He then executes one play of the first game, and then one play of the second game. If he wins in both games, he will have $5000(1.05) + 5000(1.05) = 10500$. He will allocate half (\$5250) to each game before alternating play again. That is, the bettor lets the winnings ride in both games, as before. If instead he lost in both games, he would have $2 * 5000(0.95) = 9500$, and would thus allocate \$4750 to each game before alternating play again (that is, let the losings ride in both games, as before). But if instead he wins in one game while losing in the other, he would have $5000(1.05)+5000(0.95) = 5250+4750 = 10000$, and would need to *reallocate* 250 *from the winning game to the losing game*, in order to maintain the 50% allocation weight in each game. This need to reallocate funds from the better performing game into the worse performing game will continue throughout play, whenever he wins in one while losing in the other.

The expected value of the fortune F_n after n plays of this two-game *rebalanced* strategy is

$$E[F_n(Rebalanced\ Diversification)] = F_0[\pi^2(1+f)+(1-\pi)^2(1-f)+2\pi(1-\pi)(1)]^n$$

$$= F_0(1.001)^{1000} = 2.717F_0 \tag{7}$$

which is the exact same, large expected gain as in the single (losing) game (2). But the median value will be radically different. To see this, compute the expected log utility of gross return per play to make the approximation

$$Median[F_n] \approx F_0 e^{[\pi^2 \log(1+f)+(1-\pi)^2 \log(1-f)+2*\pi(1-\pi)\log(1)]} \tag{8}$$

$$= F_0 e^{[.0003748*1000]} = 1.45F_0 \tag{9}$$

So rather than suffering a median loss of around 22% after 1000 plays of a single game, his alternating play of two identical games results in a median gain of approximately 45%, as long as he *continuously keeps his fortune split equally between the two games.* This is substantially better than the 20% median gain achieved from the Diversified Ride.

4 Discussion

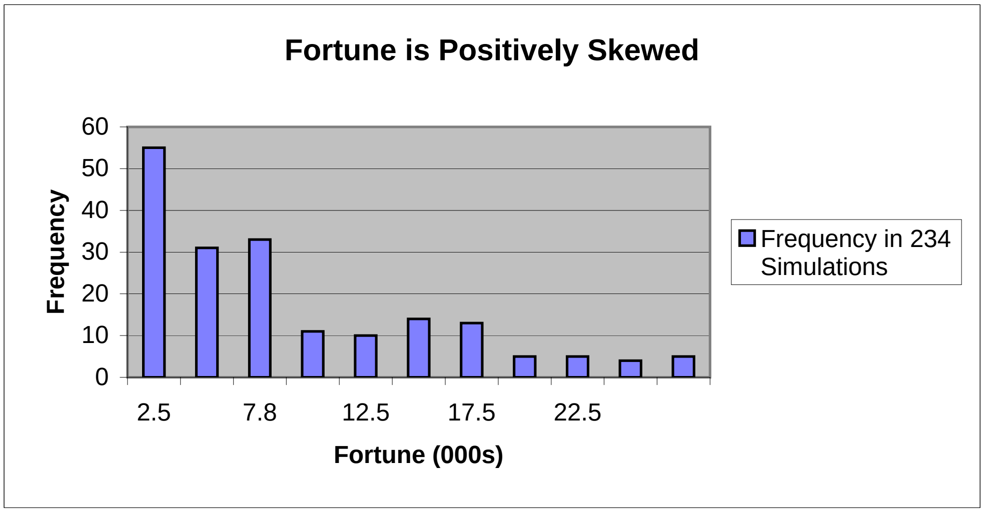
One might question whether a gambler could implement strategies like these in actual casino games. But one should not doubt that an investor could. The single game is equivalent to the *binomial tree* model of stock price evolution, which is used in standard undergraduate finance textbooks (e.g. see Hull [6, Chap.11]). Specifically, it is a model of a stock that either goes up or down by the same percentage each period (here, 5%) with complementary probabilities (51% vs. 49%), and has a positive expected net return per period (here, 10 basis points). The Diversified Ride strategy is equivalent to buying and holding an (initially) equally value-weighted portfolio of two uncorrelated (but otherwise identical) stocks. The superior Rebalanced Diversification strategy is an equally value-weighted, continuously rebalanced portfolio of the two. This provides a spectacular demonstration of the value that both buy-and-hold and continuously rebalanced diversification strategies can have for investors.

It is interesting to examine the special case of someone who wants to maximize the long-run median gain. Using the Ethier (op.cit.) approximation, one will want to find the value of f that maximizes the expected log utility of return. This is also dubbed the "log-optimal" or "Kelly" criterion [see MacLean, et.al.(op.cit.)]. In the single game with outcome (1), this will be $\arg\max_f \pi \log(1+f) + (1-\pi)\log(1-f) = 2\pi - 1$. This is $f = 2\%$ rather than 5%, and will result in a positive expected log gross return of 0.0002. As such, the single game will have a median gain, rather than the loss associated with $f = 5\%$. It is easy to show that the log-optimal bet in a favorable game (i.e. $\pi > 1/2$) will always achieve a positive expected log gross return and hence have a median gain. So that bettor cannot be the source of a Parrondo Paradox.

Finally, while one could consider policies with time-varying values of f, it is not hard to show that in i.i.d. environments, a bettor wanting to maximize expected log utility will not gain anything by doing so. Hence when using the Kelly criterion, there is no loss in generality in assuming that f is fixed across time.

5 Conclusion

The "Parrondo Paradox" that two losing games can be played in a way to produce a winning outcome is not just a probabilistic curiosity. A simplified model of Blackjack, used for analytic purposes by Ed Thorp (the famous inventor of card-counting Blackjack technique) and other analysts, was used to produce a Parrondo Paradox. Moreover, our demonstration of this used very elementary methods, including simulations that can be quickly performed by those with only basic expertise in the use of computer spreadsheets, in contrast to the more complex analyses of other Parrondo Paradoxes that abound in the literature. Those wishing to extend the conceptual framework used here should see Maslov and Zhang [9],who studied investment in an asset whose median price drifted downward over time. They constructed a rebalanced portfolio of that asset and cash, with median portfolio value drifting upward over time.



F_{1000}	Frequency	Cumulative %
2500	55	24%
5000	31	37%
7780	33	51%
10000	11	56%
12500	10	60%
15000	14	66%
17500	13	71%
20000	5	74%
22500	5	76%
25000	4	77%
27170	5	79%
More	48	100%

FIGURE 1: Histogram from 234 Simulations of F_{1000}, starting with F_0 = 10,000.

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