The Paradox of Diversification

Michael Stutzer, Professor of Finance

419 UCB, University of Colorado, Boulder, CO 80309-0419

303 492-4348

michael.stutzer@colorado.edu

Abstract

The current market malaise may keep some investors on the sidelines. The benefits of diversification may not seem as appealing in situations where the constituent investments are likely to lose money. Yet we will see, using relatively simple math, that diversification maintained by rebalancing can easily turn individual assets' negative cumulative returns into positive portfolio cumulative returns. This seemingly paradoxical result is the investing analog of a well-known phenomenon studied by physicists and mathematicians, called *Parrondo's Paradox*.

The Paradox of Diversification

1 Introduction

After reaching a high of around 1550 in October 2007, the S&P 500 is hovering around 900 as this article is written during the spring of 2009, and high market volatility around 40% has persisted. Even supposing that the inflation-adjusted expected return at this point is a comfortably positive number, e.g. 6%, the prospect is bleak.

To see why, suppose that the market evolves over time in accord with the simplest quantitative model taught in undergraduate finance texts, i.e. the binomial stock model. There are equal probabilities that the stock will either go up or down each period. When it does go up, the stock's gross return per period (say, one year) will be $1+u$, and when it goes down the gross return will be $1-d$. We must choose the model's gain percentage u and loss percentage d to match the assumed 6% expected real return and 40% volatility.[1] The statistical expected future value of the stock market, i.e. the probability-weighted average its possible future cumulative returns after n years, is the familiar calculation that compounds the market's initial value S_0 at 6% per year. To wit, after a typical long-term investment horizon of, say, $n = 30$ years the expected value is

$$E[S_n] = 1.06^n S_0 = 5.74 S_0 \tag{1}$$

which is almost six times the original investment. But this statistic is extremely misleading. The median future cumulative return, i.e. the number which has a 50-50 chance of being exceeded after 30 years, will be much lower. In fact, the median cumulative return after $n = 30$ years is

$$Median[S_n] = e^{-.019n} S_0 = e^{-.57} S_0 = 0.567 S_0 \tag{2}$$

i.e. the median outcome is a *loss* of around 43% of the initial investment. Thus there is a greater chance of losing money than there is of making anything, much less 6% per year! [2]

What accounts for this difference? The expected value is a good indicator of a likely outcome only when the probability distribution of outcomes is symmetrically bell-shaped. But because one can't lose more than the initial investment S_0, the distribution of the future cumulative return is positively skewed, rather than symmetrically bell-shaped. So the median cumulative return (here, less than 60 cents left after investing a dollar 30 years earlier) will always be lower than the (so-called!) expected cumulative return ($5.74). The difference here is particularly dramatic, and is illustrated in the top table accompanying Figure 1, containing the probabilities of possible outcomes.[3]

In light of this dismal finding, suppose someone who has read an undergraduate finance text proposes diversification into the "risk-free" asset, which we will approximate by Treasury Bills. For the sake of argument, let us assume that this asset has a constant real annual return of -10 BP.[4] Let us also assume that half the funds are initially invested at that real interest rate, and then rebalanced annually to maintain the 50-50 split with the market. With its negative inflation-adjusted return, how could diversifying into T-Bills help make money over a long horizon?

The portfolio's expected real return per year is $\frac{1}{2}6\% - \frac{1}{2}0.1\%$,or +2.95% per year, which is a bit less than half of the (losing) market's expected real return. Moreover, when the market goes up, the *portfolio's* gross return will be $1 + \frac{.46-.001}{2} = 1.2295$, gaining only about 23% while the market rose 46%. This doesn't look promising. But when the market goes down, the portfolio's gross return will be $1 - \frac{.34+.001}{2} = 0.8295$, losing only around 17% while the market loses 34%. So a string of bad years won't dent the portfolio nearly as much as it does it the market. As a result, the median of the portfolio's cumulative returns will be higher than the market's median cumulative return. Specifically, the resulting median value of the annually rebalanced portfolio after $n = 30$ years is

$$Median[Port_n] = e^{0.00984n}\, Port_0 \approx 1.343\, Port_0. \tag{3}$$

In contrast to the market's median cumulative loss of 43%, the median cumulative return from maintaining an equally value-weighted portfolio of the market and the (losing) risk-free asset is a *gain* of 34%! The lower table accompanying Figure 1 contains the more favorable probabilities of possible outcomes, which are compared with the stock in Figure 1.[5]

In summary, while separate investment in either the market or the risk-free real rate was more likely to lose real income than to earn it, rebalancing to maintain an equally-weighted portfolio of the two was more likely to earn real income than to lose it.

2 Discussion

There is nothing unusual about the expected return and volatility assumptions behind this calculation, which indeed could characterize the situation prevailing during 2009 when this was written. While we hope the US stock market volatility will decline, emerging market index portfolios do sustain such high volatility, as do many individual US stocks. Hence the example is easily generalized to other assets, or to portfolios of more than two assets. The seemingly paradoxical transformation of a bunch of losers into a winning rebalanced portfolio is very possible.

More generally, this type of phenomenon occurs in the physical sciences and in other types of decision making problems, where it is dubbed *Parrondo's Paradox.* The Wikipedia entry for Parrondo's Paradox [6] defines it as:

> Given two games, each with a higher probability of losing than winning, it is possible

> to construct a winning strategy by playing the games alternately.

The entry also includes the following assertion:

> It is of little use in most practical situations e.g. investing in stock markets, as the paradox specifically requires the payoff from at least one of the interacting games to depend on the player's capital. This is unrealistic, and would constitute a free lunch for an observant gambler if it did indeed exist.

Contrary to the above claim, we have seen that this seemingly paradoxical phenomenon can quite reasonably occur in the simplest possible model of stock market investment. Using the above terminology, the annual payoff from both the market and T-Bill "games" obviously *does* depend on the investor's capital (i.e. it is the beginning-of-year capital times the investment's gross return by year's end). Moreover, the investment example also shows that "playing the games alternately" is not critical to the result, although one can interpret the example this way. One could first issue a trading order to invest half the available funds in the market, then issue a trading order to invest the other half in T-Bills. At the end of the year, one would pool the results and do this again.

2.1 What About Buy-And-Hold Diversification?

While we have seen the power of diversification maintained by regular rebalancing, one might conjecture that something similar would happen if the manager just splits the money between the market and the fixed rate asset at the start, and then does nothing after. The median cumulative return after n years is easy to compute: it is just one-half the (losing!) median cumulative return of the market, plus one-half the non-random, losing cumulative return from the asset with fixed negative, inflation-adjusted return. So in this example, buy-and-hold diversification does not produce a paradoxical result. But the calculation would be much more complicated if the second asset also had a random real return, rather than the assumed fixed annual real return of -10BP. So the above calculation does not imply that buy-and-hold diversification could never produce a Parrondo Paradox.

3 Conclusion

A simple, yet reasonable example illustrates that rebalanced portfolio diversification can turn individually, money-losing assets into a winning portfolio. More complex and artificial examples from physics and game theory are known as *Parrondo Paradoxes*; the investment example herein is the a particularly simple Parrondo Paradox. In this example, diversifying initally and then holding (rather than regularly rebalancing back to the initial weights) does not produce this paradox – the two losing assets combine into a portfolio that still loses. While it is possible to construct more complex examples where buy-and-hold diversification does exhibit the paradox, in this regard it is less powerful than rebalanced diversification. This provides a novel illustration of the benefits of rebalancing to maintain the initial degree of diversification.

$ After 30 Years:	Probability	Cumulative
0.25	29%	29%
0.5	14%	43%
1	14%	57%
2	14%	71%
4	11%	82%
8	8%	90%
16	5%	95%
32	3%	98%

Positively Skewed Results From Investing S_0 = $1.00 In the Market For 30 Years. Losses Are More Likely Than Gains.

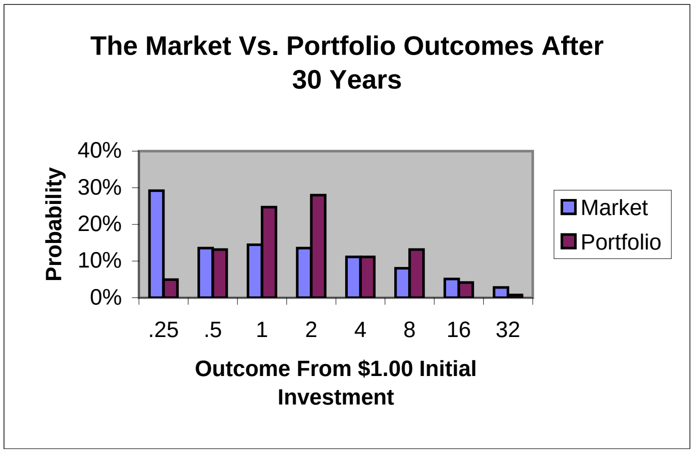

Figure 1: Comparing Skewed Probability Distributions Of Market vs. Portfolio Outcomes After 30 Years.

$ After 30 Years:	Probability	Cumulative
0.25	5%	5%
0.5	13%	18%
1	25%	43%
2	28%	71%
4	11%	82%
8	13%	95%
16	4%	99%
32	1%	100%

Results From Investing $1.00 in an Equally Weighted, Rebalanced Portfolio of the Market and an Asset That Loses a Constant 10bp/yr. to Inflation. Gains Are More Likely Than Losses.

Notes

[1]The expected gross return is $\frac{1}{2}(1+u)+\frac{1}{2}(1-d) = 1.06$ while the volatility $\sqrt{\frac{1}{2}(1+u)^2 + \frac{1}{2}(1-d)^2 - 1.06^2)} = 0.40$. Solving these two equations in the two unknowns yields the unique solution $u = 0.46$ and $d = 0.34$.

[2]Here is the derivation of (1) and (2). The market's value after n years, denoted S_n, will be

$$\begin{aligned} S_n &= S_0[(1+u)^w(1-d)^{n-w}] \\ &= S_0 e^{\log[(1+u)^w(1-d)^{n-w}]} \\ &= S_0 e^{w\log(1+u)+(n-w)\log(1-d)} \\ &= S_0 e^{[\frac{w}{n}\log(1+u)+\frac{n-w}{n}\log(1-d)]n} \qquad (4) \end{aligned}$$

where w is the binomially-distributed number of market "up" years. In accord with footnote 1, the example requires $u = 0.46$ and $d = 0.34$. Because the binomial model is IID, its mathematical expected value after n years is the compounded value of its expected gross return per year, i.e. $E[S_n] = [1.46/2 + 0.66/2]^n S_0 = 1.06^n S_0$, which is exponentially greater than the initial investment S_0. This is expression (1) in the text.

Now the median number of up years is $w_{median} = \frac{n}{2}$. Because S_n is a monotone increasing transformation of w, the median operator can be passed through to the inside of expression (4) above to find

$$\begin{aligned} Median[S_n] &= S_0 e^{\left[\frac{n/2}{n}\log(1+u)+\frac{n-n/2}{n}\log(1-d)\right]n} \\ &= S_0 e^{\left[\frac{1}{2}\log(1.46)+\frac{1}{2}\log(0.66)\right]n} \\ &= S_0 e^{-.019n} \qquad (5) \end{aligned}$$

which is exponentially less than the initial investment S_0. This is expression (2) in the text. Readers who remain unconvinced may easily simulate numerous time-paths of the binomial stock model using one spreadsheet column per time-path. For a typical long-term investment horizon, say, $n = 30$ years, spreadsheet simulated time-paths for $S_1, \ldots, S_n$ that end in a cumulative loss (i.e. $S_n < S_0$) are quite common, while time-paths ending in a value S_n that equals or exceeds the statistical expected value $1.06^{30}S_0 \approx 5.74S_0$ are very rare.

[3]Methods of forecasting future returns should also take account of this skewness; as analyzed in Hughson, Stutzer, and Yung [1].

[4]The assumption that the real return is constant is an approximation to reality. In defense of this approximation, note that while there is nominal volatility in returns generated by rolling T-Bills over, the ex-ante inflation-adjusted returns are much less volatile.

[5]The proof utilizes the math in (5) above. Using the portfolio's possible gross returns, calculate:

$$\begin{aligned} Median[Port_n] &= Port_0\ e^{\left[\frac{1}{2}\log(1.2295)+\frac{1}{2}\log(0.8295)\right]n} \\ &= Port_0\ e^{0.00984n} \end{aligned} \tag{6}$$

which is exponentially greater than the initial investment $Port_0$.

[6]Type "Parrondo's Paradox" into the search window at *http://www.wikipedia.org/*

References

[1] Eric Hughson, Michael Stutzer, and Chris Yung. The misuse of expected returns. *Financial Analysts Journal*, 62(6):88–96, 2006.