Entropic Derivative Security Valuation

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1 Mathematical Framework

Let x denote a (vector) of random variable(s), β a (vector) of parameter(s), and $f(x, \beta)$ a (column vector) of real-valued function(s). Let $E_P[f(x, \beta)]$ denote its expectation computed with probability measure P. For a specific value of β, define the set of probability measures:

$$\mathcal{P}(\beta) \equiv \{P : E_P[f(x, \beta)] = 0\} \tag{1}$$

which additionally are absolutely continuous with respect to a distinguished measure μ that is determined by the application. Selection of a particular probability measure in $\mathcal{P}(\beta)$ is called a *linear inverse problem*. Many applications of entropy make use of the following solution (when it exists):

$$\min_{P \in \mathcal{P}(\beta)} D(P\|\mu) \equiv \min_{P} \int \log(dP/d\mu)dP \;\; s.t. \; E_P[f(x, \beta)] = 0. \tag{2}$$

In (2), $D(P\|\mu) = \int \log(dP/d\mu)dP$ is the Kullback-Leibler measure of discrepancy between the measure P and the distinguished measure μ, a.k.a. the *relative entropy*. When μ is a discrete probability measure with H possible values, $D(P\|\mu) = \sum_{h=1}^{H} P_h \log(P_h/\mu_h)$, and it can be shown that $D \geq 0$, with equality when and only when $P_h = \mu_h$, $h = 1, \ldots, H$. When in addition $\mu_h \equiv 1/H$, i.e. the uniform distribution, the constrained minimization of $D(P\|\mu)$ is equivalent to maximization of the *Shannon entropy* $-\sum_h P_h \log P_h$.[2]

The solution to (2) is well-known [5, sec.3(A)] to have the following *Gibbs Canonical* or *Esscher Transformed* density:

$$\frac{dP(\beta)}{d\mu} = \frac{e^{-\gamma(\beta)' f(x,\beta)}}{E_\mu[e^{-\gamma(\beta)' f(x,\beta)}]}. \tag{3}$$

To compute the coefficient vector $\gamma(\beta)$ in (3), solve the following problem :

$$\gamma(\beta) = \arg\max_{\gamma} I(\beta, \gamma) \equiv -\log E_\mu[e^{-\gamma' f(x,\beta)}] \tag{4}$$

Finally, $D(P(\beta)\|\mu) = I(\beta, \gamma(\beta))$ defined above. This numerical value has a frequentist interpretation from the large deviations theory of IID processes that is quite useful in asset pricing model parameter estimation [11] and optimal portfolio choice [14].

2 Application to Derivative Security Valuation

Consider the simplest problem of option pricing: value a European call option, written on a single underlying stock that pays no dividends, whose price at expiration T-periods ahead is denoted $x(T)$. The riskless, continuously compounded gross interest rate r is constant between now and expiration. Under the familiar assumptions of complete and frictionless markets that do not admit arbitrage opportunities, there is a *risk neutral* probability measure P under which the call option's price C is the expected value of its risklessly discounted payoff at expiration, i.e.

(5) $$C = E_P\left[\max[x(T) - K, 0]/r^T\right]$$

where the risk-neutral probabilities P satisfy the *martingale constraint* $x(0) = E_P[\frac{x(T)}{r^T}]$, rewritten

(6) $$E_P[\frac{x(T)}{x(0)}/r^T - 1] = 0$$

Conventional risk-neutral pricing of options proceeds by specifying a parametric model for the risk-neutral stochastic price process of the underlying stock. Parameter values are found that make the model's computed stock prices and/or option prices close (e.g. in the least squares sense) to observed stock and/or option prices [3]. In the simplest case (the Black-Scholes model), this procedure requires an estimate of the volatility parameter, found either from past stock returns (i.e. historical volatility) or from market option prices (i.e. a best-fitting implied volatility).

But suppose one has doubts about the correct parametric model. The formalism of the previous section provides an alternative. Let the scalar function $f(x,\beta) = \frac{x(T)}{x(0)}/r^T - 1$, where $\beta = r$. The distribution μ is the forecast distribution of $x(T)$. To estimate this, one could just use a histogram of past T-period stock returns as in [13] and [16], a conditional histogram [15], or a more complex forecasting model [7]. The distribution is then substituted into (4) and solved to find the $\gamma(\beta)$ needed to estimate the density $P(\beta)$ in (3), which is required to compute the option valuation (5). It is possible to extend the approach to handle stochastic dividends and interest rates.

Another approach presumes a particular form for μ, and defines a vector $f(x,\beta)$ with ith component $\max[x(T) - K_i, 0]/r^T - C_i$, where C_i is the observed market price for a call with exercise price K_i, as in [10] and [4]. Then, (3)-(4) are used to study the nature of the measure $P(\beta)$ *implied* by those options' market prices, while (5) could be used to value options *other* than those present in f. See [2] for some theoretical and applied extensions of this approach.

Further refinements were developed by Gray, et.al [9], showing that the associated dynamic hedge (i.e. entropic hedge ratio) outperformed hedging benchmarks, and by Alcock and Carmichael [1], extending the concept to enable valuation of American options.

The entropic approaches are not necessarily inconsistent with the conventional approach. In fact, extant closed form option pricing models can be analytically derived by specification of the process generating the stock price distributions, and systematically applying the Esscher Transform (3) as in [8] and [6].

Notes

[1]The limited length of this entry precludes writing a survey describing all uses of entropy in finance and that cites all papers using it. Instead, focus is placed on its use in derivative security valuation, arguably the most relevant application for readers of this book, and citations are limited to early or illustrative papers.

[2]Perhaps the first published use of this in finance was in Osborne [12], who rationalized assumption of a lognormal stock price distribution as that maximizing Shannon entropy subject to a vector of two constraints (1), constraining the mean and variance of the observed continuously compounded returns to observed values.

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