Asset Allocation without Unobservable Parameters

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Some asset allocation advice for long-term investors is based on maximization of expected utility. Most commonly used investor utilities require measurement of a risk-aversion parameter appropriate to the particular investor. But accurate assessment of this parameter is problematic at best. Maximization of expected utility is thus not only conceptually difficult for clients to understand but also difficult to implement. Other asset allocation advice is based on minimizing the probability of falling short of a particular investor’s long-term return target or of an investable benchmark. This approach is easier to explain and implement, but it has been criticized by advocates of expected utility. These seemingly disparate criteria can be reconciled by measuring portfolio returns relative to the target (or benchmark) and then eliminating the usual assumption that the utility’s risk-aversion parameter is not also determined by maximization of expected utility. Financial advisors should not be persuaded by advocates of the usual expected-utility approach.

Advice about desirable quantitative asset allocations for long-term investors is abundant. The quantitative route to investor-specific advice requires the following three steps:

1. The advisor chooses a criterion function to optimize, which depends on some investor-specific information (e.g., attitude toward risk).
2. Historical time series of asset returns are used in computer optimization algorithms to estimate optimized portfolio asset allocations—one allocation for each possible set of investor-specific information.
3. Investor input is used to obtain the investor’s specific portfolio asset allocation.

For example, for a financial advisor who relies on modern portfolio theory (MPT), the first step is to choose the mean–variance criterion function, \( \mu - \gamma \sigma^2 \), where \( \mu \) denotes the mean, \( \sigma^2 \) denotes the variance of a portfolio’s return distribution, and \( \gamma \) is a parameter that is intended to measure the investor’s aversion to high variance of returns (which the theory assumes to be the appropriate measure of investor risk). In the second step, historical time series of asset returns supply the values of \( \mu \) and \( \sigma^2 \), which the advisor then uses in an optimization algorithm to estimate portfolio asset allocations along the mean–variance-efficient frontier. Finally, the advisor must specify a value of risk-aversion parameter \( \gamma \), which is required to recommend a specific efficient portfolio asset allocation. Investor input is used to determine the appropriate value of \( \gamma \).

Although quantitative financial software can provide scientifically valid assistance with Steps 1 and 2, computer software for a similarly valid implementation of Step 3 is hard to locate. For example, Siegel (2002) produced a 200-year time series of real (i.e., inflation-adjusted) asset returns to estimate mean-variance-efficient frontiers at various possible investor horizons and provided a table listing the recommended stock allocations associated with four values of \( \gamma \). He categorized the risk tolerance associated with these four values as “ultraconservative,” “conservative,” “moderate,” and “risk taking.” This approach is commonly used to implement Steps 1 and 2 of the asset allocation advice process. But Siegel was silent about the critical Step 3. How is an advisor to determine the exact value of \( \gamma \) suitable for a particular investor? An ad hoc assignment of specific ranges of \( \gamma \) to the four risk-tolerance categories does not solve the problem of assigning a particular investor to one of those categories.

Moreover, it is by no means obvious that the mean–variance criterion is always the appropriate way to implement Step 1. Siegel wrote:

> The focus of every long-term investor should be the growth of purchasing power—monetary wealth adjusted for the effect of inflation. (p. 11)
But remember that the growth rate of purchasing power—that is, inflation-adjusted, real wealth—is a random variable. If the long-term investor's sole focus is the maximum extremely long term (i.e., only asymptotically realized) growth rate of real wealth, Step 1 of the process should specify the expected growth rate of wealth as the criterion rather than Siegel's mean–variance criterion. The excellent survey by Hakansson and Ziemba (1995) noted that the expected-growth-rate-of-wealth criterion is equivalent to the expected-log-utility criterion and has been advocated by many as a suitable criterion for long-term asset allocation (e.g., Thorp 1975). Moreover, if the investor has some concerns about higher-order moments of the growth rate of real wealth, the criterion chosen in Step 1 could be the expected constant relative risk-aversion (CRRA) utility of wealth at the end of the investor's horizon (see Equation 2 or Problem 3 in the following section), which is probably the most widely used alternative utility function. The log utility is a special case of CRRA utility that is produced by a limiting lower value for its curvature parameter, γ. To illustrate this conventional method of asset allocation, I will later use Siegel's dataset to apply expected-CRRA-utility maximization to a simple but illustrative asset allocation problem.

Step 3 of the process still requires accurate assessment of a CRRA investor's risk-aversion parameter, however, which will determine what is conventionally defined to be the investor's coefficient or degree of relative risk aversion, 1 + γ. Best practice for determining this parameter will be discussed later, and I will show that it is highly problematic.

Moreover, two thoughtful and esteemed leaders in the growing field of behavioral finance, Rabin and Thaler (2001), stated that any method to measure a coefficient of relative risk aversion is doomed to failure. They concluded:

Indeed, the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of relative risk aversion, so it is a waste of time to try to measure it. (p. 225)

Thus, another asset allocation criterion—one that does not require assessment of an individual's risk-aversion parameter—is examined, namely, minimizing the probability of falling short of an investor's expressly desired target return on real wealth. Olsen (1997) provided evidence that fund managers try to avoid falling short of their expressly stated benchmarks (a goal that may sometimes be forced on managers by those who hire them). For other investors, Olsen and Khaki (1998), citing Yates's (1992) behavioral findings, asserted:

Much recent empirical evidence suggests that perceived risk is primarily a function of loss and the possibility of realizing a return below some target or aspiration level. (p. 58)

Reichenstein (1986) cited other behavioral research supporting this conception of risk. Shortfall (or its complement, outperformance) probability has also been incorporated into quantitative asset allocation models (e.g., Leibowitz and Henriksson 1989; Leibowitz and Langetieg 1989; Leibowitz, Bader, and Kogelman 1996; Browne 1999). In addition, it has been the basis for analyses of the extensively debated time diversification issue (Milevsky 1999).

So, I will discuss implementing Step 1 of the process with target-shortfall-probability minimization (or target-outperformance-probability maximization). This criterion does not require assessment of an investor's risk-aversion parameter, but it does require identification of the investor's target return. Motivated by Siegel's advice to focus on the long-term growth of real wealth, I will illustrate specifically how to use shortfall minimization to implement the three-step asset allocation process. I will then reconcile the seemingly disparate asset allocation criteria of expected CRRA utility and minimizing (maximizing) the probability of falling short of (exceeding) an investor's target or benchmark. Finally, I will reexamine the criticisms of other uses of shortfall probability lodged by influential advocates of expected utility and argue that these criticisms do not apply to the target-shortfall-probability criterion I describe.

**Conventional Use of CRRA Utility**

The criterion function chosen in Step 1 of this asset allocation process is maximization of the expected CRRA utility of end-of-holding-period inflation-adjusted wealth. To motivate the computer optimization algorithm used in Step 2 of the process, a short mathematical study of the problem is required. Let \( R_{p,t} \) denote the gross real return (1 plus the net real return) from holding a portfolio with a vector of asset allocation (value-weighted) proportions or weights denoted by \( p_{i} \) between periods \( t - 1 \) and \( t \). The real market value of the portfolio at the end of \( H \) periods is the real invested wealth, \( W_{H} \):

\[
W_{H} = W_{0} \prod_{t=1}^{H} R_{p,t}.
\]

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The expected CRRA utility of wealth at the end of $H$ periods, $E[U(W_H)]$, is:

$$E[U(W_H)] = E\left[U\left(W_0 \prod_{t=1}^{H} R_{p,t}\right)\right]$$

$$= E\left[-\left(W_0 \prod_{t=1}^{H} R_{p,t}\right)^{-\gamma}\right]$$

$$= -(W_0)^{-\gamma}E\left[\prod_{t=1}^{H} R_{p,t}\right]^{-\gamma},$$

(2)

where $\gamma$ is the investor’s risk-aversion parameter, which determines the investor’s constant coefficient of relative risk aversion, and $W_0$ is the investor’s initial wealth.

Now consider the problem of choosing possibly-time-varying portfolio asset allocation weight vectors, $p_1, p_2, \ldots, p_H$, to maximize Equation 2. Inspection of Equation 2 immediately shows that the solution will not depend on the investor’s initial wealth, $W_0$, because the variable part of Equation 2 is premultiplied by the same constant, $W_0$. Accordingly, we can set $W_0 = 1$ without loss of generality.

To proceed, we must make assumptions about the nature of the joint returns of the assets used to form the portfolio returns. The simplest asset allocation problem, which I use as the basis for both the (strictly illustrative) quantitative example in this article and the common advice to keep portfolios rebalanced to constant allocation weights, arises when the joint assets’ return process is independently and identically distributed (IID) across time. When returns are IID, the problem of maximizing Equation 2 by choosing possibly-time-varying allocation weight vectors reduces to the problem of choosing a single weight vector to maximize the following single-period expected CRRA utility:

$$\max_p E(-R_p^{-\gamma}).$$

(3)

In summary, the possibly-time-varying asset allocation weights that maximize the expected CRRA utility of real wealth at the end of the holding period (Equation 2) will not depend on the investor’s initial wealth. When the assets’ joint returns are IID across time, Equation 2 will be maximized by initially choosing a vector of value-weighted optimal-asset-allocation proportions or weights that maximizes Problem 3 and then rebalancing the portfolio at the beginning of each subsequent period back to those initial weights. The italics are to emphasize that simply buying and holding a fixed stock portfolio until the end of period $H$ is not optimal; rather, each asset’s proportion of total portfolio value must be kept fixed. Thus, some funds will always have to be moved from assets that have recently done relatively well to assets that have recently done relatively worse. One has to sell some assets “high” in order to buy some other assets “low.”

Because no one knows for certain what the exact asset return distribution is, Step 2 of the asset allocation process often proceeds by following Kroll, Levy, and Markowitz (1984) in using $T$ past periods’ historical returns to maximize the following historical-time-average estimator of Problem 3:

$$\max_{p} \frac{1}{T} \sum_{t=1}^{T} -R_{p,t}^{-\gamma}.$$  

(4)

The output of Step 2 is an optimal set of portfolio asset allocations, one for each positive value of $\gamma$.

The third (and final) step of the asset allocation process attempts to obtain an accurate value for the investor’s risk-aversion parameter, which will enable recommendation of a specific asset allocation for that investor. Later, I discuss a published questionnaire that has been used for this purpose and illustrate its implications for the simplest asset allocation problem, described here.

The simplest asset allocation problem is to recommend a fractional weight, $p$, to invest in a single risky asset (e.g., a domestic stock index) with the rest of the portfolio $(1 - p)$ invested at an ex ante constant real interest rate. Siegel argued that for U.S. investors, U.S. Treasury Inflation-Indexed Securities (commonly called TIPS) provide a vehicle for the “rest” of the assets and assumed (for simplicity in his Figure 2-7) that TIPS can be used to earn a constant real rate of 3.5 percent a year. To relate my results to his, I also adopt this assumption in the numerical examples. Strictly for pedagogical purposes, I also assume that the IID annual return process for the risky asset has the familiar binomial form frequently used in teaching: The stock index is as likely to go up, with a real gross total rate of return of $u > 1$ a year, as to go down, with a real gross total return of $d < 1$ a year.

Siegel’s 200-year historical time series of real stock returns has an (arithmetic) average annual real return of 8.27 percent (resulting in an inflation-adjusted equity premium of 8.27 percent – 3.5 percent = 4.77 percent), with a standard deviation of 18.18 percent. For this illustration, I chose a value for $u$ of 1.2645 and for $d$ of 0.9009 to match the statistics of Siegel’s data. Therefore, the asset allocation problem (Problem 3) for this example is
The \( y \)-dependent solutions of Problem 5 for stock allocation weight \( p \) are presented in Table 1.

Without using the information that the stock return process is binomial, the previous argument was that Problem 4 can be used to produce Table 1. Substituting Siegel's 200 years of historical inflation-adjusted stock returns into Problem 4 and numerically maximizing produced results that are fairly close to the exact binomial-process results from Problem 5 shown in Table 1.

Table 1 indicates that the recommended stock allocation depends critically on a relatively precise estimate of the investor's risk aversion to determine the investor's coefficient of relative risk aversion, \( 1 + y \). Increasing the degree of relative risk aversion from 2 to 4, for example, decreases the recommended stock allocation from 79 percent of portfolio value to 39 percent, with significant implications for the mean and standard deviation of the portfolio's real returns. Low values of \( 1 + y \) lead to recommended stock allocations greater than 100 percent, which is predicated on the feasibility of investing all of the investor's wealth in stock and borrowing at the (assumed constant) TIPS real interest rate to invest more (or shorting TIPS) until the end of the holding period. Because this strategy is unlikely in practice, the focus from now on will be on stock allocation fractions below 100 percent.

Table 1 also shows that the means of the expected-utility-maximizing portfolios' real returns are directly related to their volatilities, as would also be the case for mean–variance utility-maximizing portfolios in realistic problems with more than one risky asset.

### Table 1. Conventional CRRA-Utility Maximizing: Asset Allocation and Performance

<table>
<thead>
<tr>
<th>CRRA</th>
<th>Stock Weight, ( p )</th>
<th>Mean Return</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>15%</td>
<td>4.2%</td>
<td>2.8%</td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td>4.4</td>
<td>3.5</td>
</tr>
<tr>
<td>6</td>
<td>26</td>
<td>4.7</td>
<td>4.7</td>
</tr>
<tr>
<td>4</td>
<td>39</td>
<td>5.1</td>
<td>6.2</td>
</tr>
<tr>
<td>3</td>
<td>52</td>
<td>6.0</td>
<td>9.5</td>
</tr>
<tr>
<td>2</td>
<td>79</td>
<td>7.3</td>
<td>14.3</td>
</tr>
<tr>
<td>1.5</td>
<td>106</td>
<td>8.5</td>
<td>19.2</td>
</tr>
<tr>
<td>1 (log utility)</td>
<td>160</td>
<td>11.1</td>
<td>29.1</td>
</tr>
</tbody>
</table>

The sensitivity of the asset allocation recommendation to seemingly small changes in curvature parameter \( \gamma \) underscores the need for Step 3 of the asset allocation process: The advisor must estimate an accurate value of \( \gamma \) appropriate for a particular investor. In the questionnaire intended to measure an individual's \( \gamma \) value that was devised by Barsky, Juster, Kimball, and Shapiro (1997), one of the key questions is as follows:

Suppose that you are the only income earner in the family, and you have a good job guaranteed to give you your current (family) income every year for life. You are given the opportunity to take a new and equally good job, with a 50-50 chance it will double your (family) income, and a 50-50 chance that it will cut your (family) income by a third. Would you take the new job? (p. 540)

A respondent's answers to this and similar questions, in conjunction with calculations involving expected utility (Problem 3), permit the questioner to estimate an interval containing the respondent’s coefficient of relative risk aversion. For example, a respondent who answers “no” to the quoted question and another that substitutes “20 percent” for “a third” have a value of \( 1 + \gamma \) that exceeds 3.76 (Barsky et al., Table I). Barsky et al. polled more than 11,000 respondents between the ages of 51 and 61. About two out of every three respondents answered “no” to the question (Barsky et al., Table IIA); the authors estimated that the average respondent had a coefficient of relative risk aversion close to 12 (Table XI).

Hanna, Gutter, and Fan (2001) made slight modifications to the questions, to make them more appropriate to long-run (i.e., retirement) investment planning, and gave a Web-based questionnaire to 390 younger respondents. The authors still found that more than two out of three respondents had a coefficient of relative risk aversion above 3.76; their average respondent had a coefficient value close to 8.

From Table 1, these experimental estimates indicate that better than two out of every three investors should not allocate more than 39 percent of portfolio wealth to stocks, with the rest allocated to TIPS.

**Shortfall-Probability Minimization**

An investor who wants to earn an inflation-adjusted return of at least 4.5 percent a year, which is only 100 bps higher than the TIPS return assumed in the example, would probably prefer a substantially higher stock allocation than 39 percent. To see why, suppose Figure 1 is shown and explained to this investor. Figure 1 shows the familiar decreasing shortfall-probability curves for underperforming...
Figure 1. Shortfall Probabilities: Target Real Rate of Return of 4.5 Percent

<table>
<thead>
<tr>
<th>Shortfall Probability (%)</th>
<th>Holding Period (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>73</td>
<td>73</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Specifically, one curve in Figure 1 plots the horizon-dependent probability that a particular rebalanced asset allocation's cumulative return will fall short of a hypothetical account that grows at a constant gross (1 plus its net) inflation-adjusted rate of return of $r = 1.045$ a year. Figure 1 shows that an investor who wants to minimize the probability of falling short of this particular target wealth path needs to invest more than 39 percent in stocks. An 80 percent stock allocation has a shortfall probability of 21 percent at a 25-year horizon, with the probability falling to 16 percent at a 50-year horizon. (Later, I show that a stock allocation weight slightly higher than the plotted 80 percent will minimize the long-run shortfall probability.) Both probabilities are (relatively) significantly lower than would be achieved by a 39 percent allocation to stocks. Hence, the odds are about 5:1 (i.e., about a 5/6 versus 1/6 chance) that an 80 percent stock allocation will result in a growth rate of wealth exceeding 4.5 percent a year over a 50-year holding period, which accumulates to a target inflation-adjusted wealth of $1.045^{50} = $9.03 for every $1.00 initially invested.

Data sets dominated by smaller-scale investment opportunities are likely to yield much higher estimates of risk aversion than data sets dominated by larger-scale investment opportunities. Indeed, the correct conclusion to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of risk aversion, so it is a waste of time to try to measure it. (pp. 224-225)

Moreover, advice based on conventional CRRA long-run asset allocation cannot be “saved” by increasing the scale of the risks in the questionnaire. As noted, conventional CRRA expected-utility theory implies that changes in the scale of risks posed should not change the risk-aversion parameter as estimated from the questionnaire; if such changes to the questions do result in lower estimates of $\gamma$, the underlying conventional CRRA hypothesis is wrong.

Another questionable area is the reliability of individual respondents’ answers on the questionnaires. For example, if an individual were to retake the questionnaire later, the answers might be different, so the interval estimate for the individual’s $\gamma$ would be different. Simultaneously administering another questionnaire whose choices had been other growth rate higher than the TIPS rate of 3.5 percent, even though the 80 percent stock allocation will result in only a 10 percent probability of underperforming TIPS at a 50-year horizon.
seemingly innocuously altered (e.g., changing the probabilities or sizes of the gains or losses in a way that would permit similar interval $\gamma$ estimates to be made) might also yield a different interval estimate for $\gamma$.

The two survey studies cited here did not address this issue, but Yook and Everett (2003) submitted six different qualitative risk-tolerance questionnaires to 113 part-time MBA students at Johns Hopkins University. The questionnaires were used by some major brokerages and mutual funds to provide asset allocation advice for clients. Yook and Everett standardized the risk-tolerance information provided by the questionnaires in a way that assigned the number 0 to the least risk-tolerant respondent of each questionnaire and the number 100 to the most risk tolerant. If different questionnaires provided similar information about respondents’ risk tolerances, the authors expected the 0–100 scores for the respondents to be highly correlated. Yet, the 15 possible pairwise comparisons among the six questionnaires yielded an average pairwise Pearson correlation coefficient of only 56 percent. As a result, the authors concluded: 12

the 0.56 average correlation coefficient is much lower than what we should expect it to be to warrant the use of the questionnaire method without qualm. This low correlation may manifest the artificiality inherent in the risk-questionnaire design. (Yook and Everett, text above Table 1)

Moreover, advice based on conventional expected-utility asset allocation is not likely to be any less problematic if a different utility function is adopted. Rabin (2000) considered the use of any concave utility of wealth to be highly problematic:

The problems with assuming that risk attitudes over modest and large stakes derive from the same utility-of-wealth function relate to a longstanding debate in economics. Expected-utility theory makes a powerful prediction that economic actors don’t see an amalgamation of independent gambles as significant insurance against the risk of those gambles; they are either barely less willing or barely more willing to accept risks when clumped together than when apart. (p. 1287)

When returns are IID, wealth at the end of horizon $H$ results from the time-averaged sum (the “amalgamation”) of the independent, single-period, log gross portfolio returns (the “independent gambles”) earned beforehand. 13 The longer the horizon, the larger the “stakes.”

Eliminating the Risk-Aversion Parameter. As an alternative to the use of Table 1 in asset allocation, Step 2 of the proposed target-shortfall-

probability-minimizing long-term asset allocation produces the data in Table 2. 14 For each feasible target rate of return on real wealth (the assumption is that short selling and borrowing at the assumed inflation-adjusted TIPS rate over a long horizon are not possible), Table 2 reports the asset allocation required to minimize the suitably long-run probability of falling short of the target. In general, the higher the target return, the higher the minimum probability of a target shortfall. 15

**Table 2. Shortfall-Probability Minimizing: Asset Allocation and Performance**

<table>
<thead>
<tr>
<th>Target Return (percent)</th>
<th>Stock Weight, $p$</th>
<th>Minimum Shortfall Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>25 Years</td>
</tr>
<tr>
<td>5.0</td>
<td>101%</td>
<td>35%</td>
</tr>
<tr>
<td>4.5</td>
<td>83</td>
<td>24</td>
</tr>
<tr>
<td>4.0</td>
<td>58</td>
<td>24</td>
</tr>
<tr>
<td>3.7</td>
<td>37</td>
<td>11</td>
</tr>
<tr>
<td>3.5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

A financial advisor could use Table 2 to explain the unavoidable trade-off between feasible real return targets and the probabilities of achieving them. For example, a 5 percent target could increase each $1.00 of initially invested wealth by a real gross rate of return of 1.05 each year, which would accumulate to a whopping inflation-adjusted $1.05^50 or $11.48 over 50 years. According to Table 2, the probability-maximizing way to beat this long-run target requires an all-stock portfolio, and even that will result in a 24 percent (i.e., almost 1 in 4) chance of falling short of it after 50 years. In contrast, a 3.7 percent real return target (20 bps higher than the assumed return on TIPS) dictates a 37 percent weight in stocks and corresponds to a much lower cumulative inflation-adjusted multiple, 1.037^50 or 6.15 after 50 years. Table 2 shows that this strategy involves only a 6 percent (i.e., about 1:17) chance that the lower multiple will not be achieved (i.e., a 94 percent probability that it will be achieved).

The financial advisor should explain that the Table 2 data illustrate an unavoidable fundamental trade-off: High-growth portfolios are generally more volatile than low-growth portfolios. And although their higher growth helps them exceed a target return, their higher volatilities increase the chance that they will not reach the target.

Step 3, the final step of the asset allocation process, entails an interactive feedback session between the advisor (or a computer-based analytical system) and the investor. The investor considers the fundamental trade-offs between target returns
(or the investor’s equivalent target wealth levels) and the probabilities of falling short of them (or complementary probabilities of exceeding them) at various horizons. Based on this assessment, the investor who initially wanted a high target return might revise the return expectations downward because a lower target is associated with a lower probability of shortfall. If the investor must fund a fixed inflation-adjusted liability that is expected on some future date, the advisor can explain that the investor should also expect to invest a higher fraction of income to compensate for the lower target return. This trade-off might prompt the investor to again reconsider other feasible targets, savings fractions, and so on, until a satisfactory return target is established. Finally, the investor is advised to select the optimal asset allocation associated with the satisfactory return target.

Don’t put the cart before the horse. An advisor who adopts a conventional γ-dependent mean–variance or other expected-utility criterion in Step 1 of the asset allocation process might argue that explanations of the trade-offs shown in Table 2 and/or the mean–variance statistics in Table 1 can be used to help the financial advisor measure an investor’s γ. For example, an investor who receives the explanation of Table 2 might decide to adopt a target real rate of return of 4.0 percent and thus invest 58 percent in stocks. Table 1 indicates that an investor who wishes to maximize expected CRRA utility and who has a (supposedly) constant coefficient of risk aversion that is slightly less than 3 would also choose to allocate 58 percent to stocks. Hence, the financial advisor might be tempted to infer that 3 is “the” value of γ that can reliably be used to recommend an asset allocation for this investor. Similarly, a financial advisor who adopts a mean–variance utility might be tempted to present the trade-off information in Table 2 to an investor to “reverse engineer” a value for the appropriate mean–variance utility’s required risk-aversion parameter.

In realistic situations, however (that is, when multiple risky asset classes are available), an advisor’s attempts to use shortfall probability as auxiliary information in Step 3 may lead to results that are internally inconsistent with the expected-utility or mean–variance criterion adopted in Step 1. To see why, suppose there is one other asset class in addition to domestic stocks and TIPS—for example, an international stock index fund. To elicit a value for the investor’s target real rate of return, a financial advisor could expand Table 2 to include international stock and use the table to explain the asset allocation trade-offs. For illustration, assume this explanation has been made and that the corresponding recommendation to minimize target shortfall probability is to invest 50 percent in domestic stocks, 20 percent in international stocks, and the rest (30 percent) in TIPS. The required three-asset allocation analogous to Table 1 will probably not show even a single value of $1 + \gamma$ that would lead by conventional expected-CRRA-utility maximization to these three specific percentages.

The attempt to reverse engineer a γ value in this example would fail, but this failure should not be mourned, because it prevents the financial advisor from placing the cart (Step 3) before the horse (Step 1). The recommended asset allocation must be internally consistent with the optimization criterion used to produce that allocation. Investors may well find shortfall (or outperformance) probabilities to be useful and informative because they rationally want to minimize (maximize) the target shortfall (outperformance) probabilities instead of being forced to use the expected-utility or mean–variance criterion that the advisor adopted in Step 1.

For example, that mean–variance investors care only about the mean and variance of their future wealth and that they maximize the criterion $\mu - \gamma \sigma^2$ are axiomatic. A mathematical theoretical implication of this is that the investor, to select a specific combination of mean and variance and specific asset allocation that will generate it, needs to know only the mean–variance wealth trade-offs on the efficient frontier permitted by the investment opportunity set. For that investor to also need to know various target shortfall or outperformance probabilities instead of being forced to use the expected-utility or mean–variance criterion that the advisor adopted in Step 1.

Yet, even legendary mean–variance-analysis advocate William Sharpe recommends the use of the Financial Engines website, which contains the following example of how an investor can use its probabilistic assessments:

For example, a person with a 40 percent chance of achieving an income goal of $50,000 may find that tolerable if they have a 95 percent chance of achieving an income of $40,000 (in other words, their long-term downside income is $40,000). Unless portfolio returns are either normally distributed or from some other two-moment family of distributions, the quote provides more information than can be obtained from the mean and variance associated with the return distribution of the asset allocation being analyzed. Thus, although the criterion inherent in the quote does not exactly state the shortfall (outperformance) probability criterion developed in this article, it certainly is similarly motivated.
Reconciliation of Methods

The previous section illustrated how a financial advisor might use Table 2, in conjunction with input from an investor client, to recommend a specific long-run shortfall-probability-minimizing asset allocation. How were those target return-dependent asset allocations found? A theoretical extension of the results in Stutzer (2000, 2003) shows that the optimal long-term asset allocations can be found by solving the following problem:

$$\max_{p} \max_{\gamma} E \left[ \left( \frac{R_p}{r} \right)^\gamma \right].$$

(6)

Problem 6 differs from the conventional use of CRRA utility (Problem 3) by the presence of the target (gross) return per year, r, as well as the inner maximization over \( \gamma \) prior to maximizing over stock weight \( p \). For the \( r = 1.045 \) gross return target used to produce Figure 1, Problem 6 is the following simple modification of Problem 5:

$$\max_{p} \max_{\gamma} \frac{1}{2} \left[ \frac{1.2645p + 1.035(1-p)}{1.045} \right]^\gamma$$

$$+ \frac{1}{2} \left[ \frac{0.9009p + 1.035(1-p)}{1.045} \right]^\gamma.$$

(7)

Numerically solving Problem 7 in a spreadsheet yields \( p = 82.7 \) percent, as indicated in Table 2 (to the nearest percentage), which is close to the 80 percent allocation used for illustrative purposes in Figure 1. The optimized value of \( \gamma \) associated with the optimal \( p = 82.7 \) percent is approximately 0.9, but this should not be taken to imply that the investor with a 4.5 percent target return has a constant coefficient of relative risk aversion of 1.9. Through the inner maximization over \( \gamma \) in Problem 7, the investor uses a different (inner-maximizing) value of \( \gamma \) to evaluate each specific \( p \). So, although the 4.5 percent target investor does use a value of \( \gamma \approx 0.9 \) when evaluating the expected utility of a \( p = 82.7 \) percent allocation, this same investor uses an inner-maximized value of \( \gamma \approx 1.3 \) when evaluating the alternative (suboptimal) \( p = 39 \) percent allocation used to produce Figure 1. Simply put, an investor ranks the desirability of the various asset allocations by using the following function of \( p \):

$$\max_{\gamma} E \left[ \left( \frac{R_p}{r} \right)^\gamma \right] = \max_{\gamma} \frac{1}{2} \left[ \frac{1.2645p + 1.035(1-p)}{1.045} \right]^\gamma$$

$$+ \frac{1}{2} \left[ \frac{0.9009p + 1.035(1-p)}{1.045} \right]^\gamma.$$

(8)

The other long-term allocations in Table 2 were produced by successively substituting each alternative target gross return for the number 1.045 in Problem 7 and then solving it by using the maximizing routine in a spreadsheet.

In summary, conventional expected-CRRA-utility maximization and long-run target-shortfall-probability minimization can be reconciled by eliminating the conventional assumption that expected utility should be only partially maximized (i.e., maximized only over the possible asset allocations), rather than totally maximized over both the asset allocations and the positive values of \( \gamma \). This unconventional use of expected-utility theory (conventional expected-utility maximization holds the risk-aversion parameter fixed) is an application of theoretical results proven in Stutzer (2000, 2003). An analyst who (mis)specifies a conventional power utility function for a long-run investor who is seeking to minimize a target shortfall probability should find that the utility function's risk-aversion "parameter" is not a fixed constant but, instead, a variable that depends on the investor's target and the investment opportunity set.

Modifications for Realistic Situations

Some modifications of the computations are recommended for the realistic applications faced by practicing advisors. First, the numerical example dealt with only two assets, one of which was assumed to result in a constant real return (i.e., a constant real interest rate). In practice, advisors will consider a diverse group of assets, none of which earn a constant real rate of return. But despite the diversity of available asset classes, advisors should strive to avoid recommendations involving a large number of asset classes. Because of the inherent uncertainty in the use of historical return series, an analyst considering a large number of asset classes will be forced to make overly specific and possibly small percentage recommendations (e.g., invest 3 percent in South Korean stocks, 2 percent in Colorado real estate investment trusts, etc.). Such specifics convey an unrealistic degree of precision.

Second, the example investor's target was a purely hypothetical account growing at a fixed real rate of return. The advisor should make sure the investor understands that such an account is not an investable benchmark and, therefore, cannot be matched with certainty. In practice, however, investors might designate an investable benchmark (e.g., an S&P 500 Index mutual fund or exchange-traded fund), which can be matched by investing 100 percent in it. In either case, the advisor needs to find the asset allocation with the best chance of actually beating the noninvestable target or investable benchmark.
When the assets’ returns are jointly IID, only simple modifications to Problem 6 are needed. With the $i$th asset’s real return in historical period $t$ denoted by $R_{it}$, an $N$-asset portfolio’s real return with constantly rebalanced asset (value-weighted) proportions $w_1, \ldots, w_N$ is denoted

$$R_{Pt} = \sum_{n=1}^{N} w_n(R_{nt} - R_{Nt}) + R_{Nt}. \quad (9)$$

If an investable benchmark is designated, the advisor denotes its return in historical period $t$ by $R_{b,t}$. After entering the historical data in a spreadsheet or computer program, the advisor simply uses the numerical maximizer to solve the following simple generalization of Problem 4:

$$\max_{w_1, \ldots, w_N, \gamma} \max_{\gamma} \left( \frac{1}{T} \sum_{t=1}^{T} \left( \frac{R_{Pt}}{R_{b,t}} \right)^{-\gamma} \right). \quad (10)$$

That is, the advisor substitutes the designated benchmark return, $R_{b,t}$, for $r$ in Problem 6 and then uses the historical average to estimate the expected value in that formula. As in using any portfolio optimizer, the advisor can constrain the weights in Problem 10 to be nonnegative if short selling is precluded. A problem is that numerical solutions may be extreme when highly correlated assets are present. This may also happen when mean–variance-optimizing software is used. This problem provides another reason to restrict attention to a limited number of broad, disparate asset classes whose returns are not highly correlated with one another.

Also, to ensure that a numerical solution to Problem 10 does indeed exist, a rebalanced portfolio of the assets needs to be available that, when given enough time, can beat the investor’s target or designated benchmark portfolio. Thus, the rebalanced portfolio’s expected log gross return should exceed the target or benchmark’s expected log return (Stutzer 2003). In practice, with historical return data subject to these provisos, a numerical solution to Problem 10 has been easily obtained by using a spreadsheet optimizer. A valid solution finds a positive value for $\gamma$.

The assumption that returns are IID may not approximate the actual situation well enough, however, for Problem 10 to be applied with confidence that the recommendations will yield the best allocation. For example, Siegel’s (pp. 38–39) examination of stocks’ historical real returns led him to assume that future stock returns would be mean reverting rather than IID. He thus concluded that advisors should recommend considerably higher stock allocations than would otherwise be the case. Fortunately, Problem 10 can be modified to accommodate non-IID returns (Foster and Stutzer 2002). The required formula still involves maximizing over all possible values of a risk-aversion parameter, but it is more complicated than Problem 10. Instead, advisors can estimate asset allocation–dependent shortfall-probability curves (like those in Figure 1) directly. To do so, one first fixes asset allocation weights (i.e., a specific rebalanced portfolio); then, one applies a straightforward technique of bootstrapping with moving blocks (e.g., see Hansson and Persson 2000) to simulate numerous future return scenarios for the portfolio and the designated benchmark. From the scenarios, one tabulates the fraction of times the portfolio’s cumulative return at horizon $H$ falls short of the designated benchmark’s cumulative return. This fraction is the estimate of the portfolio’s shortfall probability at horizon $H$. By repeating this procedure for other portfolios (i.e., other asset allocation weights), a computer program searches for the specific asset allocation weights with the lowest shortfall probability at $H$. Long-run investors should adopt an asset allocation that minimizes the shortfall probabilities for suitably large values of $H$.

Reexamining the Arguments

Given the behavioral evidence favoring target shortfall criteria and the implementation advantages of shortfall-probability minimization, a reexamination of the typical arguments made in favor of the conventional use of expected utility and against the use of shortfall probability will be useful.

The normative case for conventional use of expected utility is grounded in the Von Neumann–Morgenstern (1980) axioms for decision making in risky situations. Von Neumann and Morgenstern started from the postulate that a decision maker is able to rank-order the desirability (i.e., from most desirable to least desirable) of different probability distributions of wealth (“wealth lotteries”) resulting from the various feasible decisions. They posed seemingly sensible axioms that decision makers might adhere to when composing this rank order. They proved that a decision maker acting in accord with those axioms acts as if he or she had adopted some utility function and had then rank-ordered the probability distributions in accord with the size of their respective expected utilities. Hence, the decision maker’s top-ranked decision should be the one that leads to the distribution of wealth with the highest expected utility.

The problem with this and other axiomatic “rationalizations” for expected-utility maximization is that axioms that appear to be reasonable on first examination do not always remain so after
closer examination. For example, much reconsideration has been given to Von Neumann and Morgenstern’s crucial “independence axiom,” without which there can be no conventional expected utility. Machina (1987) summarized the evidence showing that most individuals’ choices violate this axiom and hence cannot be consistent with expected-utility maximization. Moreover, Rabin and Rabin and Thaler pointed out the paradoxical behavior toward differently scaled risks by anyone who does maximize an expected concave utility—such as Problem 3 with $\gamma > 0$, as in the example I discussed.

In light of these negative findings about expected-utility maximization, the case against the use of shortfall probabilities is not compelling. Much of it appears to be associated with early arguments made by Samuelson (1963). The following reexamination of Samuelson’s arguments against shortfall probability will show that they either are overstated or do not apply to the criterion of minimizing target shortfall probability.

Samuelson correctly noted that in repeated betting situations analogous to long-run asset allocation, even though the probability of falling short of some wealth target (which could be mere wealth preservation) is low, “the improbable loss will be very great indeed if it does occur” (p. 110). Although technically correct, this statement might lead readers to believe that minimizing the shortfall probability will induce excessively risky behavior (presumably, by failing to adequately weigh the prospect of great but improbable shortfalls). But Samuelson also noted the obvious point that the probability of outperformance is the complement of the shortfall probability, so minimizing shortfall probability is the same as maximizing outperformance probability. Hence, readers might just as well believe that a criterion of target-outperformance-probability maximization is flawed because the improbable gain will be very great indeed if it does occur, inducing excessively conservative behavior!

The truth is that neither claim is relevant. What determines the risk borne by a shortfall-probability-minimizing (equivalently, an outperformance-probability-maximizing) investor is the specific target the investor wants to beat and the investment opportunity set the investor uses to try to beat it. The illustrative asset allocation example summarized in Table 2 shows that the full range of risk behavior is possible. To prove that not all target-shortfall-probability minimizers should prefer an 80 percent stock allocation to a 39 percent stock allocation depicted in Figure 1, Figure 2 depicts the same evaluation for a more conservative investor, one who is willing to accept a 3.7 percent target real growth rate, which is 80 bps below the target adopted by the investor using Figure 1 and only a mere 20 bps above the (assumed completely risk-free) TIPS rate of 3.5 percent. Figure 2 shows that this more conservative investor would prefer investing 39 percent in stocks to investing 80 percent in stocks.

Moreover, Samuelson’s expected-concave-utility criterion is not a reasonable alternative. This method evaluates each decision in accord with the size of the probability-weighted average of a fixed concave utility function’s values at each possible wealth outcome. The concavity yields diminishing marginal utility increments for successively larger wealth outcomes. Although this procedure may appear to be more reasonable than focusing on the probability of falling short of some target, keep in mind that it is the source of the highly problematic and paradoxical behavior toward differently scaled risks identified by Rabin and Rabin and Thaler.

Samuelson’s other criticism of outperformance probability applies only to criteria that he formulated in the following way:

the ordering principle of selecting between two actions in terms of which has the greater probability of producing a higher result does not even possess the property of being transitive. (Samuelson 1969, p. 246)

In the context of asset allocation, Samuelson is worried about the following possibility. Suppose an investor decides to prefer portfolio p’s asset allocation to that of portfolio $p'$ if there is a better than 50 percent chance (i.e., better than even odds) that the value of portfolio $p$ at the end of, say, $H = 40$ years will exceed that of portfolio $p'$. The investor then uses the same criterion to decide whether
portfolio \( p' \) is better than a third portfolio \( p'' \). Hypothetical examples can be constructed in which an investor will prefer portfolio \( p \) to portfolio \( p' \) and prefer portfolio \( p' \) to portfolio \( p'' \) but will prefer portfolio \( p'' \) to portfolio \( p \). This "intransitivity" is widely held to be irrational, and rightly so. But it cannot occur when using the criterion formulated here.

To see how it cannot occur with shortfall minimization as I have described it, suppose that at some specific horizon \( H \), portfolio \( p \) has a higher probability of outperforming an account growing at a target gross rate of return than portfolio \( p' \) has (for instance, 70 percent for portfolio \( p \) versus 60 percent for portfolio \( p' \)). And suppose that portfolio \( p'' \) has a higher probability of outperforming this target than portfolio \( p'' \) has (say, 60 percent versus 55 percent). Then, portfolio \( p \) obviously has a higher probability of outperforming the target than portfolio \( p'' \) has (i.e., 70 percent versus 55 percent). Hence, transitivity will hold no matter what \( H \) is, and the analysis has already showed that an investor who wants to maximize the long-run probability of outperforming a target gross return of, say, \( r = 1.045 \) should assign a rank order to each portfolio \( p \) in accord with the size of Problem 8, which is a function of \( p \). A rank order produced in accord with the size of any function's value is automatically transitive.

From this analysis, one can quickly see that the problem with Samuelson's formulation of an outperformance-probability criterion is its lack of a benchmark (i.e., an account growing at rate \( r \) or, alternatively, an investable portfolio of risky assets) to measure outperformance probabilities against. Samuelson's critique was apparently motivated largely by probabilistic arguments reputedly used by some advocates for expected-growth-rate maximization (i.e., the expected-log-utility criterion, not target-based or benchmark-based outperformance probability).

A third possible criticism of shortfall probability may be grounded in the work of Rubinstein (1991) and of Browne, among others, who found that significant probabilities of underperforming investor targets can persist for a long, long time—especially when maximizing the expected growth rate of wealth (i.e., expected log utility). Figure 1 illustrates the nature of this persistence in the simple asset allocation problem. But because an 80 percent stock allocation is close to the allocation that minimizes the long-run probability of falling short of the ambitious \( r = 1.045 \) growth target, Figure 1 also shows that the persistence of shortfall probabilities is unavoidable when the target is ambitious; moreover, it would only be worse if some other criterion function were used to choose a stock allocation.

Figure 2 shows that the persistence is lower when target \( r \) is lower. Generally, the solution to Problem 6 selects the asset allocation that minimizes the long-run probability of falling short of the investor's target gross return, so the shortfall probabilities will persist even longer when a different asset allocation is selected by a different criterion function.

In summary, possible intransitivities and persistence of shortfall probabilities might be used to critique arguments advocating that all investors should maximize the mean growth rate of wealth (i.e., maximize expected log utility). But neither argument provides a relevant critique of target-shortfall-probability minimization as described here.

**Conclusions**

A financial advisor's prescriptive use of either mean-variance or expected-utility-maximizing asset allocation is hampered by several difficulties. One is determining a value for the required risk-aversion parameter that is appropriate for the client. Another difficulty is explaining the rationale for maximizing these criteria. And a paradox arises when the same concave utility function is used to evaluate differently scaled risks. These difficulties can be (at least partly) alleviated by prescribing an asset allocation that minimizes the long-run probability of falling short of a target return—an inflation-adjusted rate of return. The target rate of return is determined by the advisor in consultation with the investor, who has seen and understood the trade-off between possible target rates of return and the probabilities of falling short of (and of outperforming) the targets.

Surprisingly, the seemingly disparate conventional maximization of expected CRRA utility and the minimization of long-run target shortfall probability can be reconciled by completely maximizing the expected CRRA utility of the ratio of the portfolio's return to the investor's target return. This maximization requires unconventionally maximizing the expected utility by selection of both the portfolio's asset allocation weights and the utility's risk-aversion parameter (as opposed to conventional maximization over the weights alone with the use of some fixed value of the risk-aversion parameter). This unconventional formulation of minimizing long-run target shortfall probability retains the framework of expected-utility maximization while eliminating the conventional but problematic requirement that the advisor fix a value of the risk-aversion parameter that is most appropriate for the investor. Instead, in an interactive feedback process,
4. A simple proof follows. Set \( W_Q = 1 \) in Equation 2, risk-aversion-parameter-dependent expected util-
shortfall-minimization (target-outperformance-
are either overstated or not applicable to target-
benchmark the investor wants to beat.
5. In practice, rebalancing may also be a useful strategy for
maximizing other criterion functitms, even when a more
accurate floating point values were used in the computations.
Because of this discrepancy, readers may be unable to
exactly duplicate the calculations that yielded the tables
here but should get close enough to determine whether they
are doing them correctly.
6. Later, I describe how the analysis should be modified to
address realistic applications with multiple assets, more
realistic return processes, and a situation in which no asset
earns a constant real rate of return.
7. Matching the average real stock return requires the solution
of one equation in the two unknowns, \( u \) and \( d \); matching the
standard deviation requires the solution of another equa-
tion in the two unknowns. The assumed values of \( u \) and \( d \)
are the only ones satisfying both equations. They are
reported to only four decimal places in the text, but more
accurate floating point values were used in the computations.
8. Restricting investors to using stocks and non-inflation-
indexd bonds, Siegel derived a mean–variance-efficient
stock allocation of 115 percent for an investor with a 30-year
horizon and what he described as "moderate risk tolerance"
(p. 38). This high percentage occurred because his 200-year series of inflation-adjusted stock returns had volatilities typical of series generated from a mean-reverting
process rather than an IID process. As a result, the 30-year cumulative real returns presented in his series have lower
volatility than would occur if returns were IID and were thus
more favorably appraised by his mean–variance criterion
than IID returns would be. When the possibility of allocating
assets to TIPS (earning an assumed real rate of 3.5 percent a
year) was added, the mean–variance-efficient "langency"
portfolio (of only the stocks and non-inflation-indexed
bonds) for an investor with a horizon of 10 years still allo-
cated 185 percent to stocks; hence, it was 85 percent short in
bonds. A mean–variance utility-maximizing investor’s risk-
aversion parameter would then determine only the relative
split of wealth between that portfolio and TIPS. Siegel thus
concluded, “Stocks should constitute the overwhelming
proportion of all long-term financial portfolios” (p. 361).
Because my purpose was to use the simplest possible asset allocation problem to illustrate the crucial implementa-
tional differences between conventional mean–variance and
expected-utility criteria and the target-shortfall-probability
criterion, I did not use the controversial hypothesis that real
stock returns are in fact (rather than simply historically
appearing to be) generated by a mean-reverting process. But
later, I do describe the modifications required to adapt the
computations to incorporate mean reversion and/or other
apparent deviations from the example’s IID assumption.
9. Of course, implicit in Table 1 is an assumption that the
investor is concerned only about inflation-adjusted invested
wealth at one point in time (i.e., \( H \) years in the future). Siegel
made this assumption, as have many others in applications
of portfolio choice theory. Because Siegel’s focus was (and
my focus is) on asset allocation advice for long-term inves-
tors, horizon \( H \) might be thought of as the number of years
until the investor will retire or die. In the case of retirement
perhaps (and surely in the case of death), the advice should
not be too dependent on knowing the particular value of \( H \).
Fortunately, advice based on Table 1 does not depend at all
on \( H \), although such need not be the case when returns are
non-IID.

Notes
1. Because I focus solely on the asset allocation advice for long-
term investors, I do not address the controversial subject of
time diversification.
2. Moreover, any future critiques of this criterion must be
considered in light of the new and surprisingly devastating
criticisms in Rabin (2000) and Rabin and Thaler that apply
to all conventional expected-concave-utility-of-wealth crite-
3. Mossin (1968) used a more complicated dynamic program-
ning method to study this problem. A later influential
paper by Samuelson (1969) used a different criterion
function—the time-additive sum of single-period CRRA
utilities of consumption during the holding period. Mossin’s
end-of-holding-period utility of wealth did not permit
withdrawals for consumption until the end of the holding
period. Hence, the derivations in Samuelson (1969) do not
apply to the problem studied in this paper, although they
do lead to similar results.
4. A simple proof follows. Set \( W_Q = 1 \) in Equation 2, which produces the equivalent criterion function of
\(-E\left[ \prod_{i=1}^{H} R_{p_i}^{n} \right]^{-1}\). When returns are independently
distributed, we can move the expectation operator inside
the product to yield \( \prod_{i=1}^{H} E(R_{p_i}^{n}) \). We immediately see
that the portfolio weights that maximize this function will be
the weights that minimize \( \prod_{i=1}^{H} E(R_{p_i}^{n}) \). Moreover,
because the logarithmic function is monotonically increas-
ing, the same weight vectors will also minimize
log \( \prod_{i=1}^{H} E(R_{p_i}^{n}) = \sum_{i=1}^{H} \log E(R_{p_i}^{n}) \). Because this
problem is additively separable, the first-order necessary
condition for any particular \( p_i \) will not include any of the
other \( p_i \)'s. As a result, \( p_i \) can be found simply by
minimizing \( E(R_{p_i}^{n}) \). If we make use of the assumption that
portfolio returns are independently distributed across the
\( H \) equal-length periods, we can remove the subscript \( t \)
from \( p_i \) in this problem, producing \( E(R_{p_i}^{n}) \). Multiplying
\( E(R_{p_i}^{n}) \) by \(-1\) converts it into the one-period maximization
in the article (Problem 3).
5. In practice, rebalancing may also be a useful strategy for
maximizing other criterion functions, even when a more
elaborate dynamic strategy could, in theory, do better (e.g.,
when portfolio returns are not IID). Buetow, Sellers, Trotter,
Hunt, and Whipple (2002) documented some of the general
benefits of rebalancing.
6. Later, I describe how the analysis should be modified to
address realistic applications with multiple assets, more
realistic return processes, and a situation in which no asset
earns a constant real rate of return.
7. Matching the average real stock return requires the solution
of one equation in the two unknowns, \( u \) and \( d \); matching the
benefits of rebalancing.
10. For each stock weight $p_i$ let $R_i^H$ denote the portfolio's gross (1 plus net) return when the stock goes up and $R_i$ denote the portfolio's gross return when the stock goes down. For a horizon length $T$, shortfall occurs when there are $x$ or fewer up moves—that is, when $x R_i (T-x) R_i < 1.045^T$. Hence, the probability of shortfall is computed by evaluating the cumulative binomial distribution (with probability of an up move equal to 1/2) at the highest integer $x$ satisfying this inequality. This calculation can be easily accomplished in a spreadsheet.

11. Technically, some of these individuals might have revealed a coefficient of risk aversion between 3.76 and 4.00 by this single answer, although the questionnaire provides no way to tell.

12. I thank Michele Gambera of Morningstar for providing this reference. Although the six questionnaires did not attempt to estimate an investor's risk tolerance in the quantitative way described by Barsky et al. and Hanna et al., the results indicate that an analogous study of reliability is warranted, one in which different questionnaires pose qualitatively different wealth lotteries.

13. That is, $W_H = W_0 \prod_{i=1}^{n} R_i^H = W_0 e^{\sum_{i=1}^{n} \log R_i^H}$.

14. The next section describes how this table was produced and how it could be produced in more realistic problems.

15. The relationship between target return and probability in Table 2 is only weakly (rather than strictly) monotonic because of the discrete, binomial distribution process used to model the stock return. The relationship would be strictly monotonic had an absolutely continuous (e.g., lognormal) distribution been assumed in the calculations, and one would obtain a strictly increasing relationship between the target return and the (minimized) probability of falling short of it at any horizon $H$.

16. The Financial Engines website (www.financialengines.com) describes Sharpe as "Founder Bill Sharpe" and describes his role as follows: "One of the fathers of Modern Portfolio Theory, Nobel laureate Dr. Sharpe has helped some of the nation's largest pension fund managers invest billions of dollars of retirement money. When he realized that he could offer this help to individuals through an advice technology platform, he started Financial Engines. Financial Engines combines Dr. Sharpe's pioneering investment methodology with scalable technology to provide all investors with cost-effective, expert advice."

17. This sentence was found in the "Common Questions" section of www.financialengines.com as part of the answer to the question: "What is a good Forecast?" In the quote, the "Forecast" is the 40 percent probability of outperforming the investor's $50,000 target (i.e., "income goal").

18. Arguments in Samuelson (1963) were repeated in numerous later articles; see Samuelson 1969; Merton and Samuelson 1974; Samuelson 1979).

References


