CHAPTER 8

Index Models

Advantages of the Single Index Model

- Reduces the number of inputs for diversification
- Easier for security analysts to specialize

Single Factor Model

$$r_i = E(r_i) + \beta_i m + e_i$$

β_i response of an individual security's return to the common factor *m*. Beta measures systematic risk.

m a common macroeconomic factor that affects all security returns. The S&P 500 is often used as a proxy for *m*.

e_i firm-specific surprises

Single-Index Model

- Regression Equation:

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

- The expectation of the residual term e_i is zero, so the expected return-beta relationship is:

$$E(R_i) = \alpha_i + \beta_i E(R_M)$$

Single-Index Model

Risk and covariance:

- Variance - Systematic risk and Firm-specific risk, assume noise is uncorrelated:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$$

- Covariance - product of betas x market index risk:

$$Cov(r_i, r_j) = \beta_i \beta_j \sigma_M^2$$

Single-Index Model - Correlation

Product of correlations with the market index:

$$Corr_{i,j}\left(r_i, r_j\right) = Cov_{i,j}\left(r_i, r_j\right) / \left(\sigma_i \sigma_j\right)$$

$$Corr_{i,j}\left(r_i, r_j\right) = \frac{\beta_i \beta_j \sigma_M^2}{\sigma_i \sigma_j} \times \frac{\sigma_M^2}{\sigma_M \sigma_M} =$$

$$\frac{\beta_i \sigma_M^2}{\sigma_i \sigma_M} \frac{\beta_j \sigma_M^2}{\sigma_j \sigma_M} = Corr\left(r_i, r_M\right) \times Corr\left(r_j, r_M\right)$$

Questions to test your intuition

- What is the stock's E(r) if $(r_M - r_f) = 0$?
- What is the responsiveness of the stock to market movements relative to r_f?
- What is the stock-specific component of return (not driven by the market)?
- What is the variance attributable to uncertainty of the market?
- And that attributable to firm-specific events?

Index Model and Diversification

- Consider an Equally weighted portfolio and take the expected return R_P as the average:

$$R_P = \frac{1}{n}\sum_{i=1}^{n} R_i = \frac{1}{n}\sum_{i=1}^{n}\left(\alpha_i + \beta_i R_M + e_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\alpha_i \quad + \frac{1}{n}\sum_{i=1}^{n}\beta_i R_M \quad + \frac{1}{n}\sum_{i=1}^{n} e_i$$

$$R_P = \quad \alpha_P \qquad + \beta_P R_M \qquad + e_P$$

Index Model and Diversification

- The portfolio variance by definition:

$$\sigma_P^2 = \beta_P^2 \sigma_M^P + \sigma^2(e_P)$$

where the market component comes from the portfolio's sensitivity to the market:

$$\beta_P = \frac{1}{n}\sum_{i=1}^{n} \beta_i$$

and the non-systemic component $\sigma^2(e_P)$ is the contribution of all the stocks in the portfolio.

Index Model and Diversification

- Variance of the non-systemic component of an equally weighted portfolio is (we assume all the stock-specific components are uncorrelated):

$$\sigma^2(e_P) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2(e_i) = \frac{1}{n}\overline{\sigma}^2(e)$$

- When *n* gets large, $\sigma^2(e_P)$ becomes negligible and firm specific risk can be diversified away.

Figure 8.1 The Variance of an Equally Weighted Portfolio with Risk Coefficient β_p

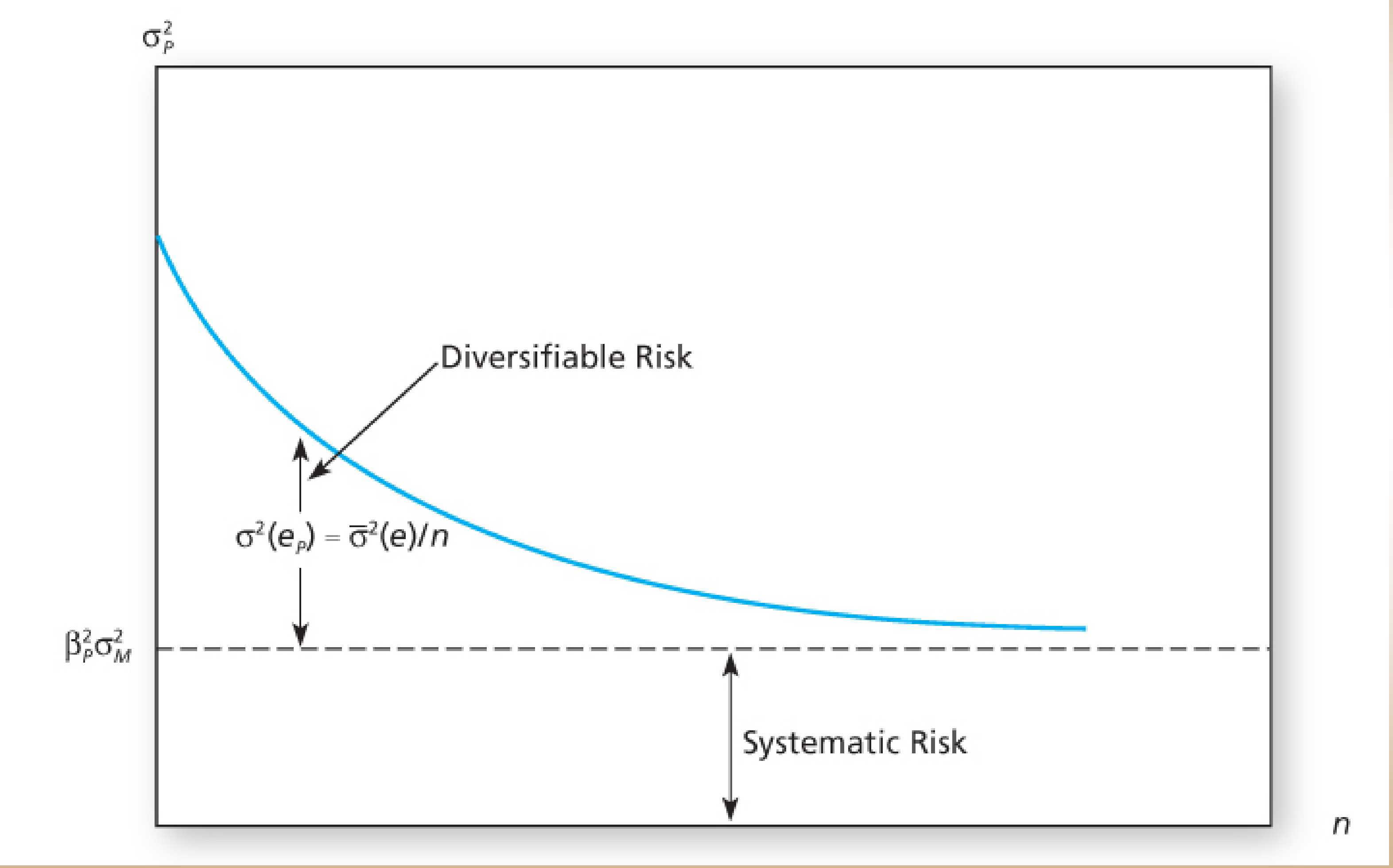


Figure 8.2 Excess Returns on HP and S&P 500

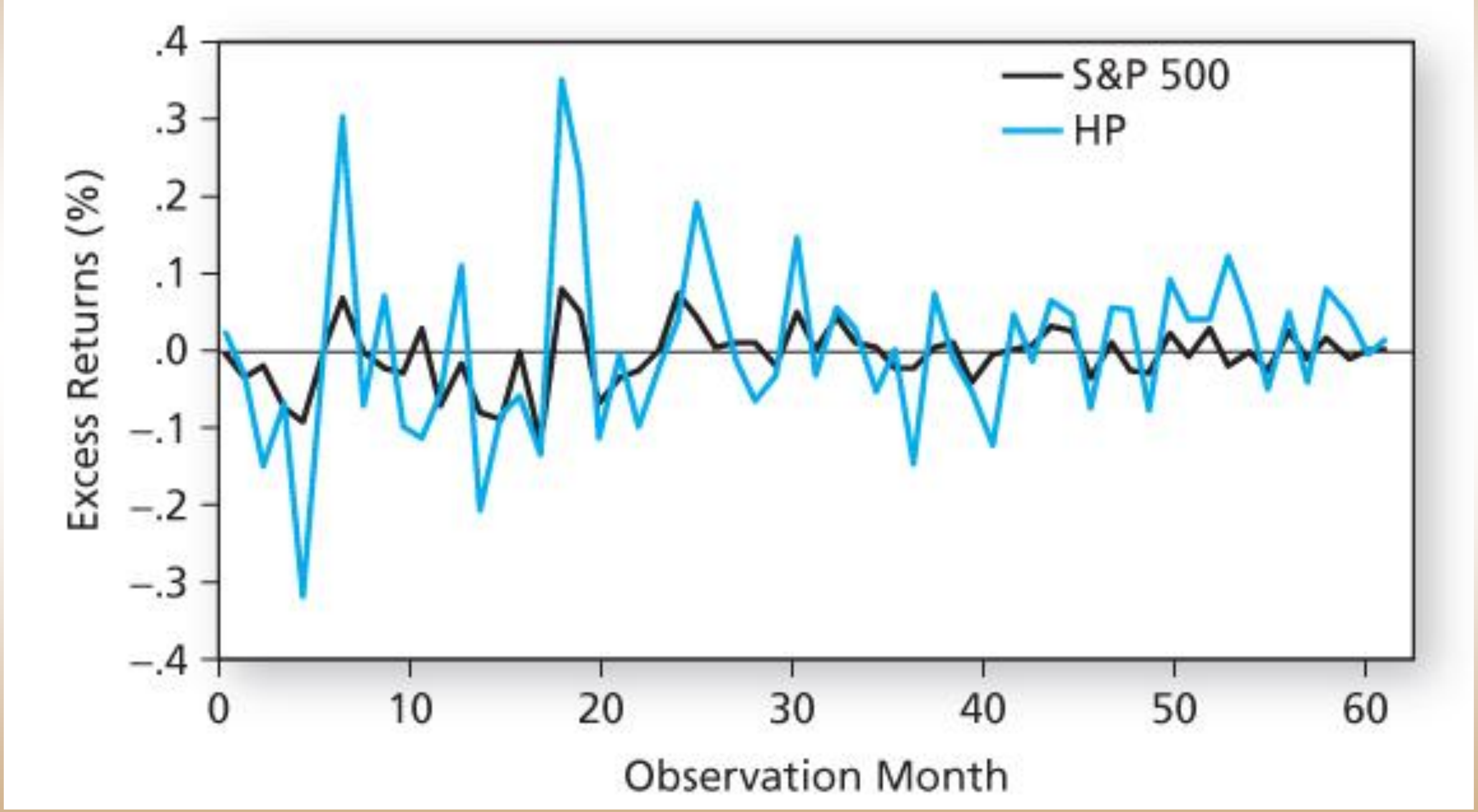


Figure 8.3 Scatter Diagram of HP, the S&P 500, and HP's Security Characteristic Line (SCL)

$$R_{HP}(t) = \alpha_{HP} + \beta_{HP} R_{SP500}(t) + e_{HP}(t)$$

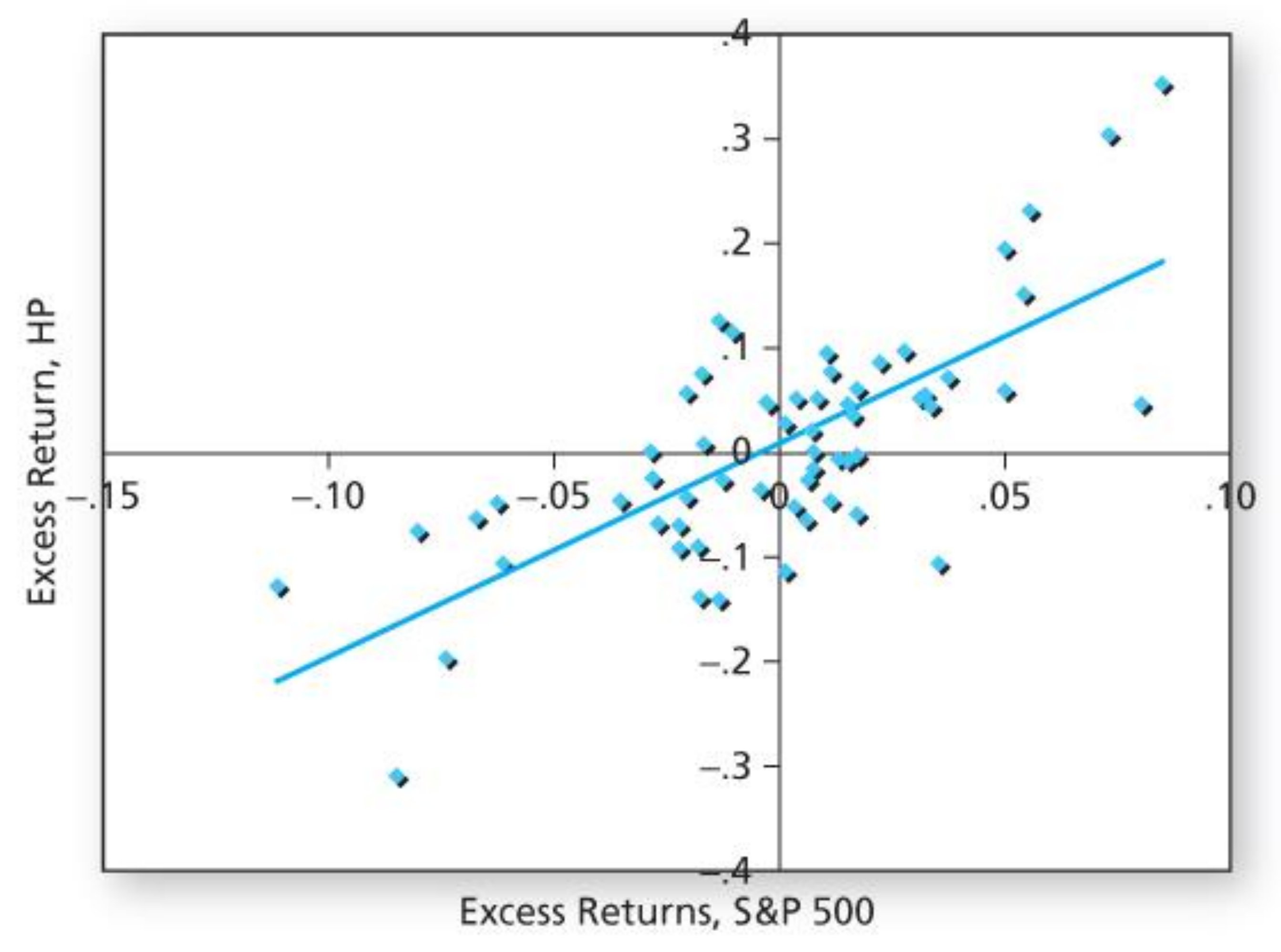


Table 8.1 Excel Output: Regression Statistics for the SCL of Hewlett-Packard

Regression Statistics		
Multiple R	.7238	← correlation
R-square	.5239	← explanatory power
Adjusted R-square	.5157	
Standard error	.0767	
Observations (monthly)	60	

ANOVA

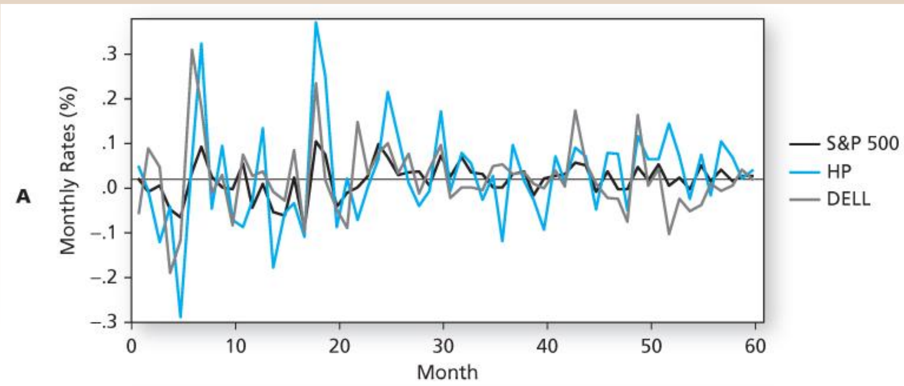
	df	SS	MS
Regression	1	.3752	.3752
Residual	58	.3410	.0059
Total	59	.7162	

	Coefficients	Standard Error	*t-Stat*	*p-Value*
Intercept	$\alpha \rightarrow$ 0.0086	.0099	0.8719	.3868
S&P 500	$\beta \rightarrow$ 2.0348	.2547	7.9888	.0000

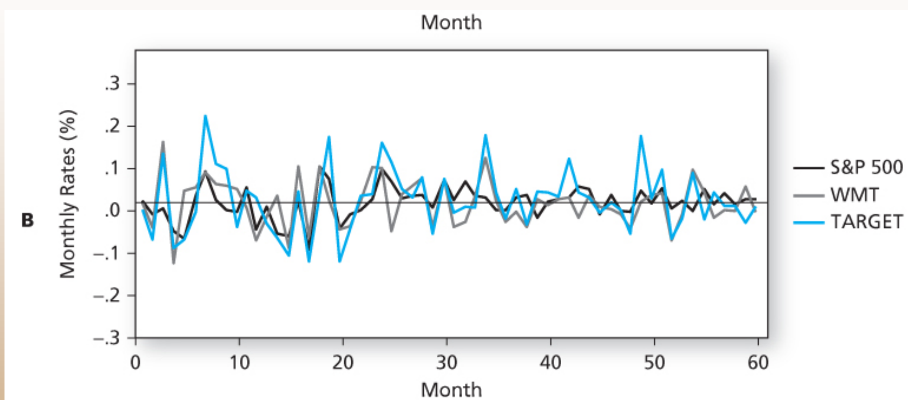
Table 8.1 Interpretation

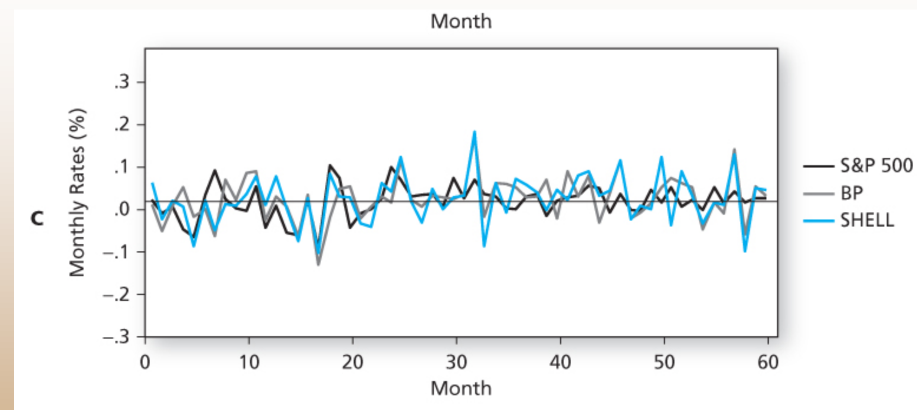
- Correlation of HP with the S&P 500 is 0.7238
- The model explains about 52% of the variation in HP
- HP's alpha is 0.86% per month (10.32% p.a.), but it is not statistically significant
- HP's beta is 2.0348, but the 95% confidence interval is +/- ~2 standard errors, which is quite wide

Figure 8.4 Excess Returns on Portfolio Assets



- Study pairs of securities vs the market to estimate correlations
- Compute stats to measure correlations





Study portfolio stats – 1

Panel 1: Risk Parameters of the Investable Universe (annualized)

	SD of Excess Return	Beta	SD of Systematic Component	SD of Residual	Correlation with the S&P 500
S&P 500	0.1358	1.00	0.1358	0	1
HP	0.3817	2.03	0.2762	0.2656	0.72
DELL	0.2901	1.23	0.1672	0.2392	0.58
WMT	0.1935	0.62	0.0841	0.1757	0.43
TARGET	0.2611	1.27	0.1720	0.1981	0.66
BP	0.1822	0.47	0.0634	0.1722	0.35
SHELL	0.1988	0.67	0.0914	0.1780	0.46

Panel 2: Correlation of Residuals

	HP	*DELL*	*WMT*	*TARGET*	*BP*
HP	1				
DELL	0.08	1			
WMT	–0.34	0.17	1		
TARGET	–0.10	0.12	0.50	1	
BP	–0.20	–0.28	–0.19	–0.13	1
SHELL	–0.06	–0.19	–0.24	–0.22	0.70

A closer look at correlations

Panel 2: Correlation of Residuals					
	HP	*DELL*	*WMT*	*TARGET*	*BP*
HP	1				
DELL	0.08	1			
WMT	–0.34	0.17	1		
TARGET	–0.10	0.12	0.50	1	
BP	–0.20	–0.28	–0.19	–0.13	1
SHELL	–0.06	–0.19	–0.24	–0.22	0.70

Study portfolio stats – 2

Panel 3: The Index Model Covariance Matrix

		S&P 500	HP	DELL	WMT	TARGET	BP	SHELL
	Beta	1.00	2.03	1.23	0.62	1.27	0.47	0.67
S&P 500	1.00	0.0184	0.0375	0.0227	0.0114	0.0234	0.0086	0.0124
HP	2.03	0.0375	0.1457	0.0462	0.0232	0.0475	0.0175	0.0253
DELL	1.23	0.0227	0.0462	0.0842	0.0141	0.0288	0.0106	0.0153
WMT	[illegible]	[illegible]	[illegible]	0.0141	0.0374	0.0145	0.0053	0.0077
TARGET	[illegible]	[illegible]	[illegible]	0.0288	0.0145	0.0682	0.0109	0.0157
BP	[illegible]	[illegible]	[illegible]	0.0106	0.0053	0.0109	0.0332	0.0058
SHELL	[illegible]	[illegible]	[illegible]	0.0153	0.0077	0.0157	0.0058	0.0395

$$\beta_i \beta_j \sigma_M^2$$

$$\beta_i^2 \sigma_M^2 + \sigma^2(e_i)$$

Cells on the diagonal (shadowed) equal to variance
formula in cell C26 = B4^2
Off-diagonal cells equal to covariance
formula in cell C27 = C$25*$B27*B4^2
multiplies beta from row and column by index variance

Panel 4: Macro Forecast and Forecasts of Alpha Values

	S&P 500	HP	DELL	WMT	TARGET	BP	SHELL
Alpha	0	0.0150	–0.0100	–0.0050	0.0075	0.012	0.0025
Risk premium	0.0600	0.1371	0.0639	0.0322	0.0835	0.0400	0.0429

Example: build optimal portfolio

Panel 5: Computation of the Optimal Risky Portfolio									
	S&P 500	*Active Pf A*	HP	DELL	WMT	TARGET	BP	SHELL	Overall Pf
$\sigma^2(e)$			0.0705	0.0572	0.0309	0.0392	0.0297	0.0317	
$\alpha/\sigma^2(e)$		0.5505	0.2126	–0.1748	–0.1619	0.1911	0.4045	0.0789	
$w^0(i)$		1.0000	0.3863	–0.3176	–0.2941	0.3472	0.7349	0.1433	
$[w^0(i)]^2$			0.1492	0.1009	0.0865	0.1205	0.5400	0.0205	
α_A		0.0222							
$\sigma^2(e_A)$		0.0404							
w_A^0		0.1691							
w^*(Risky portf)	0.8282	0.1718							
Beta	1	1.0922	2.0348	1.2315	0.6199	1.2672	0.4670	0.6736	1.0158
Risk premium	0.06	0.0878	0.1371	0.0639	0.0322	0.0835	0.0400	0.0429	0.0648
SD	0.1358	0.2497							0.1422
Sharpe ratio	0.44	0.35							0.46

Alpha and Security Analysis

1. Use **Macroeconomic analysis** to estimate risk premium and risk of the market index (R_M, σ_M)

2. Use **statistical analysis** to estimate the beta coefficients of all securities and their residual variances $\sigma^2(e_i)$

Alpha and Security Analysis

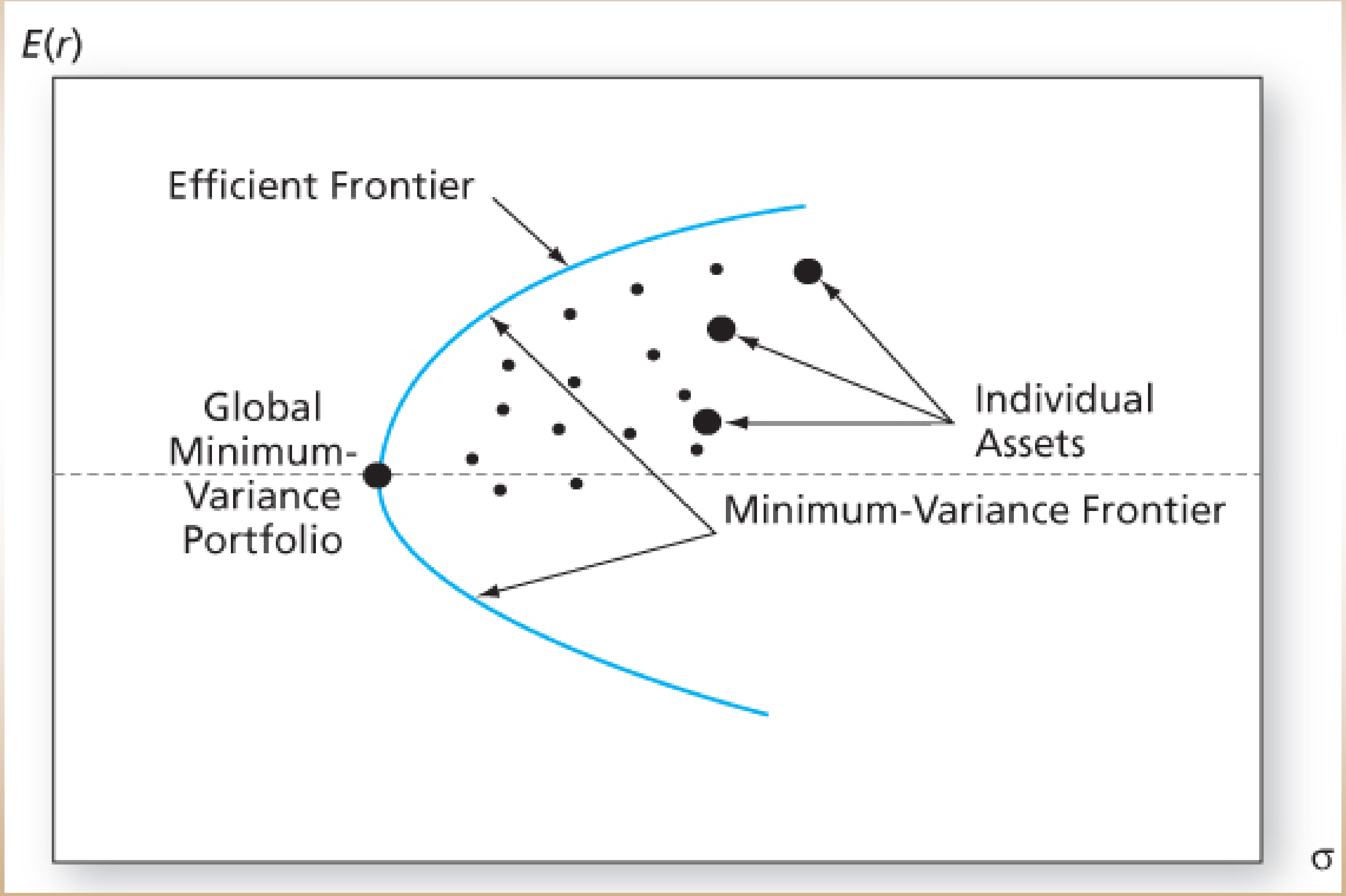
3. Use **numerical methods** to establish the expected return of each security *independently of security analysis* (β)

4. Use **security analysis** to develop your own forecast of the expected returns for each security (α)

Single-Index Model considerations

- Techniques for estimating β are well known
- Estimating alpha requires a deep knowledge of the company behind the stock:
  - Positive α means overweight in the portfolio
  - What do you do if α is negative?

Recall the Minimum-Variance Frontier



Chapter 7 took the entire universe of stocks and used brute-force math to find the efficient frontier

Single-Index Model – Optimization

- Single-Index model offers a simpler optimization than the model in chapter 7 as the model is simplified
- Include the market as asset n+1 to improve diversification. By definition:
  - Beta of market index = 1
  - Alpha of market index = 0
  - $e_{market\_index} = 0$

Single-Index Model Input List

- Risk premium on the S&P500 portfolio (R_M)
- Estimate of the SD of the S&P500 portfolio (σ_M)
- *n* sets of estimates for each stock of:
  - Beta coefficient
  - Stock residual variances
  - Alpha values

Single-Index Model steps

- Use R_M, alphas and betas to construct *n+1* expected returns
- Use betas and σ_M to construct the covariance matrix
- Set up the optimization problem to minimize portfolio variance, given a return, subject to…
- …constraint that weights add up to one
- You could use excel solver to solve this problem and build your efficient frontier

Index Model – Recall α_P and β_P

Consider a generic portfolio and take the excess return R_P as the average:

$$R_P = \sum_{i=1}^{n} w_i R_i = \sum_{i=1}^{n} w_i \left(\alpha_i + \beta_i R_M + e_i \right)$$

$$= \sum_{i=1}^{n} w_i \alpha_i + \sum_{i=1}^{n} w_i \beta_i R_M + \sum_{i=1}^{n} w_i e_i$$

$$R_P = \alpha_P + \beta_P R_M + e_P$$

Optimal Risky Portfolio of the Single-Index Model

Now take the portfolio *expected* excess return:

$$E(R_P) = \alpha_P + E(R_M)\,\beta_P$$

$$= \sum_{i=1}^{n} w_i \alpha_i + E(R_M) \sum_{i=1}^{n} w_i \beta_i$$

Optimal Risky Portfolio of the Single-Index Model

Standard Deviation and Sharpe Ratio:

$$\sigma_P^2 = \beta_P^2 \times \sigma_M^2 + \sigma^2(e_P)$$

$$= \left(\sum_{i=1}^{n} w_i \beta_i\right)^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma^2(e_i)$$

$$S_P = E(R_P) / \sigma_P$$

Optimal Risky Portfolio of the Single-Index Model

- No need to use Excel as there is an analytical solution
- Solution is a combination of:
  - –Active portfolio (*A*), with weight w_A
  - –Market-index passive portfolio (*M*)

Optimal Risky Portfolio - w_A

Assume for a moment beta=1

Then the optimal weight w_A is proportional to the ratio $\sigma_A / \sigma^2(e_A)$ to balance excess return and residual variance from Active portfolio A:

$$w_A^0 = \frac{\alpha_A / \sigma^2(e_A)}{E(R_M) / \sigma_M^2}$$

Optimal Risky Portfolio of the Single-Index Model

Next, modify of active portfolio weight w_A to optimize, as beta is not necessarily =1:

$$w_A^* = \frac{w_A^0}{1+(1-\beta_A)w_A^0}$$

Notice that when

$$\beta_A = 1 \text{ then } w_A^* = w_A^0$$

The Information Ratio

The Sharpe ratio of an optimally constructed risky portfolio will exceed that of the index portfolio (the passive strategy):

$$S_P^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma(e_A)} \right]^2$$

"Information" ratio

The Information Ratio

- The contribution of the active portfolio depends on the ratio of its alpha to its residual standard deviation.

- The information ratio measures the extra return we can obtain from security analysis.

Figure 8.5 Efficient Frontiers with the Index Model and Full-Covariance Matrix

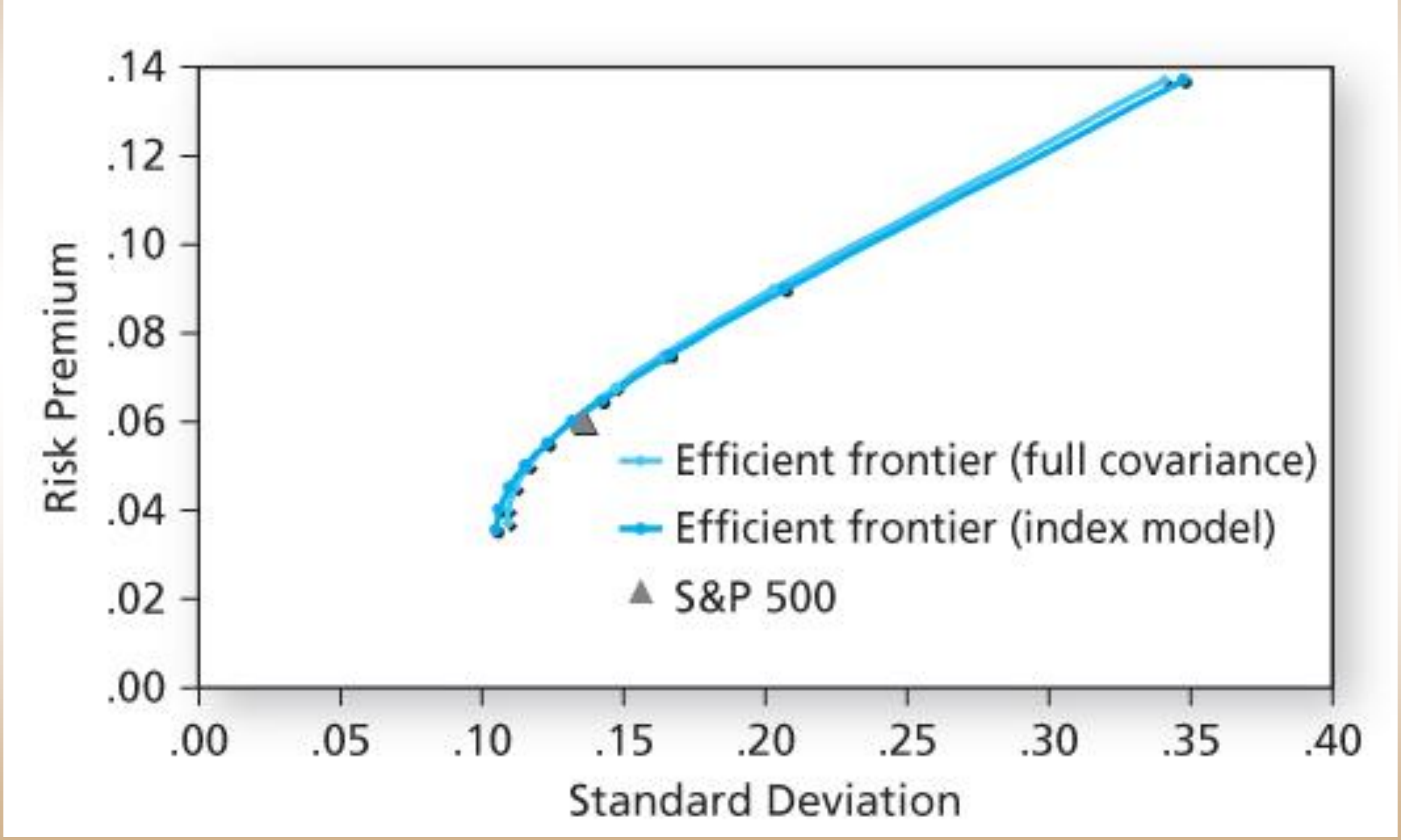


Table 8.2 Portfolios from the Single-Index and Full-Covariance Models

	Global Minimum Variance Portfolio		Optimal Portfolio	
	Full-Covariance Model	Index Model	Full-Covariance Model	Index Model
Mean	.0371	.0354	.0677	.0649
SD	.1089	.1052	.1471	.1423
Sharpe ratio	.3409	.3370	.4605	.4558
Portfolio Weights				
S&P 500	.88	.83	.75	.83
HP	−.11	−.17	.10	.07
DELL	−.01	−.05	−.04	−.06
WMT	.23	.14	−.03	−.05
TARGET	−.18	−.08	.10	.06
BP	.22	.20	.25	.13
SHELL	−.02	.12	−.12	.03

Is the Index Model Inferior to the Full-Covariance Model?

- Full Markowitz model may be better in principle, but:
  - Using the full-covariance matrix invokes estimation risk of <u>thousands</u> of terms
  - <u>Cumulative errors</u> may result in a portfolio that is actually inferior to that derived from the single-index model
  - The single-index model is <u>practical</u> and <u>decouples</u> macro and security analysis.

Beta Book: Industry Version of the Index Model

- Use 60 most recent months of price data
- Use S&P 500 as proxy for M
- Compute total returns that ignore dividends
- Estimate index model without excess returns:

$$r = a + br_m + e^*$$

Beta Book: Industry Version of the Index Model

Adjust beta because:

- The average beta over all securities is 1. Thus, our best forecast of the beta would be that it is 1.
- Also, firms may become more “typical” as they age, causing their betas to approach 1.

Table 8.4 Industry Betas and Adjustment Factors

Industry	Beta	Adjustment Factor
Agriculture	0.99	−.140
Drugs and medicine	1.14	−.099
Telephone	0.75	−.288
Energy utilities	0.60	−.237
Gold	0.36	−.827
Construction	1.27	.062
Air transport	1.80	.348
Trucking	1.31	.098
Consumer durables	1.44	.132