CHAPTER 10: ARBITRAGE PRICING THEORY AND MULTIFACTOR MODELS OF RISK AND RETURN

PROBLEM SETS

1. The revised estimate of the expected rate of return on the stock would be the old estimate plus the sum of the products of the unexpected change in each factor times the respective sensitivity coefficient:

   revised estimate = 12% + [(1 × 2%) + (0.5 × 3%)] = 15.5%

2. The APT factors must correlate with major sources of uncertainty, i.e., sources of uncertainty that are of concern to many investors. Researchers should investigate factors that correlate with uncertainty in consumption and investment opportunities. GDP, the inflation rate, and interest rates are among the factors that can be expected to determine risk premiums. In particular, industrial production (IP) is a good indicator of changes in the business cycle. Thus, IP is a candidate for a factor that is highly correlated with uncertainties that have to do with investment and consumption opportunities in the economy.

3. Any pattern of returns can be "explained" if we are free to choose an indefinitely large number of explanatory factors. If a theory of asset pricing is to have value, it must explain returns using a reasonably limited number of explanatory variables (i.e., systematic factors).

4. Equation 10.9 applies here:

   $E(r_p) = r_f + \beta_{P1}[E(r_1) - r_f] + \beta_{P2}[E(r_2) - r_f]$

   We need to find the risk premium (RP) for each of the two factors:

   $RP_1 = [E(r_1) - r_f]$ and $RP_2 = [E(r_2) - r_f]$

   In order to do so, we solve the following system of two equations with two unknowns:

   $.31 = .06 + (1.5 \times RP_1) + (2.0 \times RP_2)$

   $.27 = .06 + (2.2 \times RP_1) + [(-0.2) \times RP_2]$

   The solution to this set of equations is:

   $RP_1 = 10\%$ and $RP_2 = 5\%$

   Thus, the expected return-beta relationship is:

   $E(r_P) = 6\% + (\beta_{P1} \times 10\%) + (\beta_{P2} \times 5\%)$

5. The expected return for Portfolio F equals the risk-free rate since its beta equals 0.
   For Portfolio A, the ratio of risk premium to beta is: (12 − 6)/1.2 = 5

   For Portfolio E, the ratio is lower at: (8 – 6)/0.6 = 3.33

   This implies that an arbitrage opportunity exists. For instance, you can create a Portfolio G with beta equal to 0.6 (the same as E's) by combining Portfolio A and Portfolio F in equal weights. The expected return and beta for Portfolio G are then:

   $E(r_G) = (0.5 \times 12\%) + (0.5 \times 6\%) = 9\%$

   $\beta_G = (0.5 \times 1.2) + (0.5 \times 0\%) = 0.6$

   Comparing Portfolio G to Portfolio E, G has the same beta and higher return. Therefore, an arbitrage opportunity exists by buying Portfolio G and selling an equal amount of Portfolio E. The profit for this arbitrage will be:

$$r_G - r_E = [9\% + (0.6 \times F)] - [8\% + (0.6 \times F)] = 1\%$$

That is, 1% of the funds (long or short) in each portfolio.

6. Substituting the portfolio returns and betas in the expected return-beta relationship, we obtain two equations with two unknowns, the risk-free rate (r_f) and the factor risk premium (RP):

$$12\% = r_f + (1.2 \times RP)$$

$$9\% = r_f + (0.8 \times RP)$$

Solving these equations, we obtain:

$$r_f = 3\% \text{ and } RP = 7.5\%$$

7. a. Shorting an equally-weighted portfolio of the ten negative-alpha stocks and investing the proceeds in an equally-weighted portfolio of the ten positive-alpha stocks eliminates the market exposure and creates a zero-investment portfolio. Denoting the systematic market factor as R_M, the expected dollar return is (noting that the expectation of non-systematic risk, *e*, is zero):

$$\$1{,}000{,}000 \times [0.02 + (1.0 \times R_M)] - \$1{,}000{,}000 \times [(-0.02) + (1.0 \times R_M)]$$

$$= \$1{,}000{,}000 \times 0.04 = \$40{,}000$$

The sensitivity of the payoff of this portfolio to the market factor is zero because the exposures of the positive alpha and negative alpha stocks cancel out. (Notice that the terms involving R_M sum to zero.) Thus, the systematic component of total risk is also zero. The variance of the analyst's profit is not zero, however, since this portfolio is not well diversified.

For n = 20 stocks (i.e., long 10 stocks and short 10 stocks) the investor will have a $100,000 position (either long or short) in each stock. Net market exposure is zero, but firm-specific risk has not been fully diversified. The variance of dollar returns from the positions in the 20 stocks is:

$$20 \times [(100{,}000 \times 0.30)^2] = 18{,}000{,}000{,}000$$

The standard deviation of dollar returns is $134,164.

b. If n = 50 stocks (25 stocks long and 25 stocks short), the investor will have a $40,000 position in each stock, and the variance of dollar returns is:

$$50 \times [(40{,}000 \times 0.30)^2] = 7{,}200{,}000{,}000$$

The standard deviation of dollar returns is $84,853.

Similarly, if n = 100 stocks (50 stocks long and 50 stocks short), the investor will have a $20,000 position in each stock, and the variance of dollar returns is:

$$100 \times [(20{,}000 \times 0.30)^2] = 3{,}600{,}000{,}000$$

The standard deviation of dollar returns is $60,000.

Notice that, when the number of stocks increases by a factor of 5 (i.e., from 20 to 100), standard deviation decreases by a factor of $\sqrt{5} = 2.23607$ (from $134,164 to $60,000).

8. a. $\sigma^2 = \beta^2\sigma_M^2 + \sigma^2(e)$

$\sigma_A^2 = (0.8^2 \times 20^2) + 25^2 = 881$

$\sigma_B^2 = (1.0^2 \times 20^2) + 10^2 = 500$

$\sigma_C^2 = (1.2^2 \times 20^2) + 20^2 = 976$

b. If there are an infinite number of assets with identical characteristics, then a well-diversified portfolio of each type will have only systematic risk since the non-systematic risk will approach zero with large n:

Well-Diversified σ_A^2 □ 256

Well-Diversified σ_B^2 □ 400

Well-Diversified σ_C^2 □ 576

The mean will equal that of the individual (identical) stocks.

c. There is no arbitrage opportunity because the well-diversified portfolios all plot on the security market line (SML). Because they are fairly priced, there is no arbitrage.

9. a. A long position in a portfolio (P) comprised of Portfolios A and B will offer an expected return-beta tradeoff lying on a straight line between points A and B. Therefore, we can choose weights such that $\beta_P = \beta_C$ but with expected return higher than that of Portfolio C. Hence, combining P with a short position in C will create an arbitrage portfolio with zero investment, zero beta, and positive rate of return.

b. The argument in part (a) leads to the proposition that the coefficient of β^2 must be zero in order to preclude arbitrage opportunities.

10. a. E(r) = 6% + (1.2 × 6%) + (0.5 × 8%) + (0.3 × 3%) = 18.1%

b. Surprises in the macroeconomic factors will result in surprises in the return of the stock:

Unexpected return from macro factors =

[1.2 × (4% – 5%)] + [0.5 × (6% – 3%)] + [0.3 × (0% – 2%)] = –0.3%

E (r) =18.1% − 0.3% = 17.8%

11. The APT *required* (i.e., equilibrium) rate of return on the stock based on r_f and the factor betas is:

required E(r) = 6% + (1 × 6%) + (0.5 × 2%) + (0.75 × 4%) = 16%

According to the equation for the return on the stock, the actually expected return on the stock is 15% (because the *expected* surprises on all factors are zero by definition). Because the actually expected return based on risk is less than the equilibrium return, we conclude that the stock is overpriced.

12. The first two factors seem promising with respect to the likely impact on the firm's cost of capital. Both are macro factors that would elicit hedging demands across broad sectors of investors. The third factor, while

important to Pork Products, is a poor choice for a multifactor SML because the price of hogs is of minor importance to most investors and is therefore highly unlikely to be a priced risk factor. Better choices would focus on variables that investors in aggregate might find more important to their welfare. Examples include: inflation uncertainty, short-term interest-rate risk, energy price risk, or exchange rate risk. The important point here is that, in specifying a multifactor SML, we not confuse risk factors that are important to a particular investor with factors that are important to investors in general; only the latter are likely to command a risk premium in the capital markets.

13. The formula is: $E(r) = 0.04 + 1.25 \times 0.08 + 1.5 \times 0.02 = .17 = 17\%$

14. If $r_f = 4\%$ and based on the sensitivities to real GDP (0.75) and inflation (1.25), McCracken would calculate the expected return for the Orb Large Cap Fund to be:

$$E(r) = 0.04 + 0.75 \times 0.08 + 1.25 \times 0.02 = .04 + 0.085 = 8.5\% \text{ above the risk free rate}$$

Therefore, Kwon's fundamental analysis estimate is congruent with McCracken's APT estimate. If we assume that both Kwon and McCracken's estimates on the return of Orb's Large Cap Fund are accurate, then no arbitrage profit is possible.

15. In order to eliminate inflation, the following three equations must be solved simultaneously, where the GDP sensitivity will equal 1 in the first equation, inflation sensitivity will equal 0 in the second equation and the sum of the weights must equal 1 in the third equation.

1. $1.25wx + 0.75wy + 1.0wz = 1$
2. $1.5wz + 1.25wy + 2.0wz = 0$
3. $wx + wy + wz = 1$

Here, "x" represents Orb's "High Growth Fund", "y" represents "Large Cap Fund" and "z" represents "Utility Fund." Using algebraic manipulation will yield wx = wy = 1.6 and wz = -2.2.

16. Since retirees living off a steady income would be hurt by inflation, this portfolio would not be appropriate for them. Retirees would want a portfolio with a return positively correlated with inflation to preserve value, and less correlated with the variable growth of GDP. Thus, Stiles is wrong. McCracken is correct in that supply side macroeconomic policies are generally designed to increase output at a minimum of inflationary pressure. Increased output would mean higher GDP, which in turn would increase returns of a fund positively correlated with GDP.

17. The maximum residual variance is tied to the number of securities (n) in the portfolio because, as we increase the number of securities, we are more likely to encounter securities with larger residual variances. The starting point is to determine the practical limit on the portfolio residual standard deviation, $\sigma(e_P)$, which still qualifies as a 'well-diversified portfolio.' A reasonable approach is to compare $\sigma^2(e_P)$ to the market variance, or equivalently, to compare $\sigma(e_P)$ to the market standard deviation. Suppose we do not allow $\sigma(e_P)$ to exceed $p\sigma_M$, where p is a small decimal fraction, for example, 0.05; then, the smaller the value we choose for p, the more stringent our criterion for defining how diversified a 'well-diversified' portfolio must be.

Now construct a portfolio of n securities with weights $w_1, w_2,\dots,w_n$, so that $\Sigma w_i = 1$. The portfolio residual variance is: $\sigma^2(e_P) = \Sigma w_1^2\sigma^2(e_i)$

To meet our practical definition of sufficiently diversified, we require this residual variance to be less than $(p\sigma_M)^2$. A sure and simple way to proceed is to assume the worst, that is, assume that the residual variance of each security is the highest possible value allowed under the assumptions of the problem: $\sigma^2(e_i) = n\sigma^2_M$

In that case: $\sigma^2(e_P) = \Sigma w_i^2\, n\sigma_M^2$

Now apply the constraint: $\Sigma w_i^2\, n\, \sigma_M^2 \le (p\sigma_M)^2$

This requires that: $n\Sigma w_i^2 \le p^2$

Or, equivalently, that: $\Sigma w_i^2 \le p^2/n$

A relatively easy way to generate a set of well-diversified portfolios is to use portfolio weights that follow a geometric progression, since the computations then become relatively straightforward. Choose w_1 and a common factor q for the geometric progression such that $q < 1$. Therefore, the weight on each stock is a fraction q of the weight on the previous stock in the series. Then the sum of n terms is:

$$\Sigma w_i = w_1(1- q^n)/(1- q) = 1$$

or: $\quad w_1 = (1- q)/(1- q^n)$

The sum of the n *squared* weights is similarly obtained from w_1^2 and a common geometric progression factor of q^2. Therefore:

$$\Sigma w_i^2 = w_1^2(1- q^{2n})/(1- q^2)$$

Substituting for w_1 from above, we obtain:

$$\Sigma w_i^2 = [(1- q)^2/(1- q^n)^2] \times [(1- q^{2n})/(1- q^2)]$$

For sufficient diversification, we choose q so that: $\Sigma w_i^2 \le p^2/n$

For example, continue to assume that $p = 0.05$ and $n = 1{,}000$. If we choose $q = 0.9973$, then we will satisfy the required condition. At this value for q:

$$w_1 = 0.0029 \text{ and } w_n = 0.0029 \times 0.9973^{1,000}$$

In this case, w_1 is about 15 times w_n. Despite this significant departure from equal weighting, this portfolio is nevertheless well diversified. Any value of q between 0.9973 and 1.0 results in a well-diversified portfolio. As q gets closer to 1, the portfolio approaches equal weighting.

18. a. Assume a single-factor economy, with a factor risk premium E_M and a (large) set of well-diversified portfolios with beta β_P. Suppose we create a portfolio Z by allocating the portion w to portfolio P and (1 – w) to the market portfolio M. The rate of return on portfolio Z is:

$$R_Z = (w \times R_P) + [(1 - w) \times R_M]$$

Portfolio Z is riskless if we choose w so that $\beta_Z = 0$. This requires that:

$$\beta_Z = (w \times \beta_P) + [(1 - w) \times 1] = 0 \Rightarrow w = 1/(1 - \beta_P) \text{ and } (1 - w) = -\beta_P/(1 - \beta_P)$$

Substitute this value for w in the expression for R_Z:

$$R_Z = \{[1/(1 - \beta_P)] \times R_P\} - \{[\beta_P/(1 - \beta_P)] \times R_M\}$$

Since $\beta_Z = 0$, then, in order to avoid arbitrage, R_Z must be zero.

This implies that: $R_P = \beta_P \times R_M$

Taking expectations we have:

$$E_P = \beta_P \times E_M$$

This is the SML for well-diversified portfolios.

b. The same argument can be used to show that, in a three-factor model with factor risk premiums E_M, E_1 and E_2, in order to avoid arbitrage, we must have:

$$E_P = (\beta_{PM} \times E_M) + (\beta_{P1} \times E_1) + (\beta_{P2} \times E_2)$$

This is the SML for a three-factor economy.

19. a. The Fama-French (FF) three-factor model holds that one of the factors driving returns is firm size. An index with returns highly correlated with firm size (i.e., firm capitalization) that captures this factor is SMB (Small Minus Big), the return for a portfolio of small stocks in excess of the return for a portfolio of large stocks. The returns for a small firm will be positively correlated with SMB. Moreover, the smaller the firm, the greater its residual from the other two factors, the market portfolio and the HML portfolio, which is the return for a portfolio of high book-to-market stocks in excess of the return for a portfolio of low book-to-market stocks. Hence, the ratio of the variance of this residual to the variance of the return on SMB will be larger and, together with the higher correlation, results in a high beta on the SMB factor.

b. This question appears to point to a flaw in the FF model. The model predicts that firm size affects average returns, so that, if two firms merge into a larger firm, then the FF model predicts lower average returns for the merged firm. However, there seems to be no reason for the merged firm to underperform the returns of the component companies, assuming that the component firms were unrelated and that they will now be operated independently. We might therefore expect that the performance of the merged firm would be the same as the performance of a portfolio of the originally independent firms, but the FF model predicts that the increased firm size will result in lower average returns. Therefore, the question revolves around the behavior of returns for a portfolio of small firms, compared to the return for larger firms that result from merging those small firms into larger ones. Had past mergers of small firms into larger firms resulted, on average, in no change in the resultant larger firms' stock return characteristics (compared to the portfolio of stocks of the merged firms), the size factor in the FF model would have failed.

Perhaps the reason the size factor seems to help explain stock returns is that, when small firms become large, the characteristics of their fortunes (and hence their stock returns) change in a significant way. Put differently, stocks of large firms that result from a merger of smaller firms appear empirically to behave differently from portfolios of the smaller component firms. Specifically, the FF model predicts that the large firm will have a smaller risk premium. Notice that this development is not necessarily a bad thing for the stockholders of the smaller firms that merge. The lower risk premium may be due, in part, to the increase in value of the larger firm relative to the merged firms.

CFA PROBLEMS

1. a. This statement is incorrect. The CAPM requires a mean-variance efficient market portfolio, but APT does not.

   b. This statement is incorrect. The CAPM assumes normally distributed security returns, but APT does not.

   c. This statement is correct.

2. b. Since Portfolio X has $\beta = 1.0$, then X is the market portfolio and $E(R_M) = 16\%$. Using $E(R_M) = 16\%$ and $r_f = 8\%$, the expected return for portfolio Y is not consistent.

3. d.

4. c.

5. d.

6. c. Investors will take on as large a position as possible only if the mispricing opportunity is an arbitrage. Otherwise, considerations of risk and diversification will limit the position they attempt to take in the mispriced security.

7. d.

8. d.