Foundation-Penalty Cuts for Mixed-Integer Programs

*Fred Glover

**Hanif D. Sherali

*Leeds School of Business
University of Colorado
Boulder, CO 80309-0419
fred.glover@colorado.edu
(Corresponding author)

**Grado Department of Industrial and Systems Engineering (0118)
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061
hanifs@vt.edu

Abstract

We propose a new class of Foundation-Penalty (FP) cuts for MIPs that are easy to generate by exploiting routine penalty calculations. Their underlying concept generalizes the lifting process and provides derivations of major classical cuts. (Gomory cuts arise from low level FP cuts by simply 'plugging in' standard penalties.)

Keywords: *mixed-integer programming, cutting planes, B&B penalties*

This research was partially supported by the *Office of Naval Research* contract N00014-01-1-0917 in connection with the Hearin Center of Enterprise Science at the University of Mississippi, and by the *National Science Foundation* under Grant Number 0094462.

1. Introduction

Consider a mixed-integer zero-one program MIP stated in the form

MIP: Minimize $c^T x$

subject to $x \in X \equiv X_1 \cap X_2$ (1)

where $x \in R^n$, X_1 describes a set of constraints representing a polyhedron in R^n, $X_2 \equiv \{ x : x_j$ is binary $\forall j \in B \subseteq N \}$, and where $N=\{1,...,n\}$ is the index set of all the variables.

In this paper, we introduce the concepts behind a new class of cutting planes for MIP called *Foundation-Penalty cuts* **(FP).** Although we focus on zero-one mixed-integer programs, as discussed in the sequel, many of the ideas extend to general mixed-integer programs as well. As the name suggests, FP cuts are predicated on two elements: a (linear) foundation function, and a set of penalties that are computed based on the conditional values taken on by either a single binary variable, or by several binary variables comprising a *generalized upper bounding* **(GUB)** set. While we discuss both the single integer variable and GUB set cases, it is in the latter context of GUB sets that this class of FP cuts might hold the greatest promise. Previous work that yields special cutting planes for such GUB constrained problems can be found, for example, in Glover [8, 9], Hammer et al. [17], Balas [2], Sherali et al. [27], Sherali and Lee [26], Glover et al. [10], and Gu et al. [16].

In fact, as we demonstrate in the sequel, the concept underlying FP cuts generalizes the lifting process introduced by Gomory [14] and Glover [17] in the context of group polyhedra, and by Padberg [21]for 0-1 problems (see also Crowder et al. [6], Balas and Zemel [5], Gu et al. [15], and Nemhauser and Wolsey [20]. Moreover, the FP cuts bear a relationship to the disjunctive cuts (see Balas et al. [3. 4], Sherali and Adams [23, 24], and Sherali et al. [25], convexity cuts (see Glover [8], Gomory cuts [12, 13], and mixed-integer rounding cuts (see Marchand and Wolsey [19]). We explore these relationships in the present paper to afford insights into exploiting the flexibility that is inherent in the class of FP cuts for generating judicious types of cuts, as well as tightening other cuts that might have been derived using alternative mechanisms. In particular, this flexibility permits the derivation of valid inequalities that cut deeper along specified dimensions as desired.

The remainder of this paper is organized as follows. The next section provides the basic concept underlying the derivation of FP cuts. The relationship of this idea to the lifting process is exposed in Section 3. Certain higher-order extensions of these cuts that consider multiple, non-GUB-related, binary variables are investigated in Section 4. Thereafter, we explore the relationship of FP cuts to several other classical cuts in Section 5, and Section 6 concludes the paper with recommendations for future research in this area.

2. Derivation of the Foundation-Penalty (FP) Cuts

As mentioned in Section 1, the class of FP cuts are governed by two principal elements, namely, a foundation function, and certain penalty computations conditioned on values taken on by either a single binary variable or by a set of GUB-constrained variables. Each of these features that leads to the derivation of the cut is discussed in turn below. (Later in Section 4, we shall extend these cuts to higher-order or multiple variable disjunctions.)

2.1. Foundation Function

The foundation function is some selected linear function of the form $\sum_{j\in J} d_j x_j$, where $J \subseteq N$. Typically, this function might correspond to a reduced cost objective representation associated with some dual feasible solution, or more pertinently, an optimal basis to the linear programming (LP) relaxation $\overline{\text{MIP}}$ of MIP, given by

$$\overline{\textbf{MIP}}\textbf{:}\ \text{Minimize}\ \left\{c^T x : x \in \overline{X}\right\} \tag{2}$$

where $\overline{X}$ denotes the usual LP relaxation of X. In this case, we would have

$$J \equiv \{\text{set of nonbasic variables}\},\ \text{and}\ d_j \geq 0 \quad \forall j \in J. \tag{3}$$

In addition, a variety of other foundation functions can be used as discussed in the sequel. Also, in the context of tightening existing cuts, the foundation function might correspond to the linear functional form of a previously generated cut.

2.2 Penalty Characterizations

The penalty computations are conducted with respect to some binary variable $x_k, k \in B$, or with respect to a set of variables that are GUB-constrained according to

$$\sum_{k\in K} x_k = 1,\ \text{where}\ K \subseteq B. \tag{4}$$

(The case of multiple non-GUB restricted binary variables is considered later in Section

4, and the treatment of general integer-variables is addressed in Corollary 1 below. However, for clarity in presentation, we focus on the aforementioned two cases first.)

For any binary variable x_k, $k \in B$, let us define P_{k1} and P_{k0} as lower bounds on the respective values z_{k1} and z_{ko} of MIP under the corresponding additional conditions based on the disjunction that $x_k = 1$ or $x_k = 0$. More specifically,

$$P_{k1} \leq z_{k1} \equiv \text{minimum}\left\{\sum_{j\in J} d_j x_j : x \in X, \text{ and } x_k = 1\right\}, \text{ and} \tag{5a}$$

$$P_{k0} \leq z_{k0} \equiv \text{minimum}\left\{\sum_{j\in J} d_j x_j : x \in X, \text{ and } x_k = 0\right\}. \tag{5b}$$

Observe that while LP relaxations afford the most natural mechanism for computing these lower bounding values P_{k1} and P_{k0} in (5), there are a variety of different alternatives by which these quantities might be generated. For example, these values could be based on the simple penalties derived via a single dual simplex pivot on an optimal LP tableau for $\overline{\text{MIP}}$ that has been augmented by the additional restriction $x_k = 1$ or $x_k = 0$, or via multiple dual simplex pivots of this type as used in *strong branching* strategies (see Applegate et al. [1]). Alternatively, we could solve integer knapsack relaxations based on surrogate constraint strategies (see Rardin and Karwan [22]). Of course, if any of the penalty computations yield $P_{k1} = \infty$ or $P_{k0} = \infty$, we simply enforce the opposite restriction $x_k = 0$ or $x_k = 1$, respectively, and conduct subsequent implied reductions via standard logical tests (see Nemhauser and Wolsey [20]). Hence, in what follows, we will always assume that all penalties derived are finite. Note that in the GUB case, we compute P_{k1} for each $k \in K$, where for any $k \in K$, the computation of P_{k1} in

(5a) is conducted by also explicitly enforcing $x_j = 0\ \forall j \in K - \{k\}$ by virtue of the presence of (4) within the defining set *X*, and similarly, for other GUB sets that contain x_k. Albeit simple, this observation is frequently overlooked in the literature on MIP penalty calculations, yet can have a considerable impact on the penalties generated, particularly in problems where a given x_k belongs to numerous GUB sets. The effect of compelling the indicated accompanying GUB variables to equal zero will automatically be achieved if a penalty calculation is based on performing a sufficient number of dual pivots, but possibly at the expense of undue computational effort. Setting $x_j = 0$ can of course be conveniently handled for any nonbasic x_j simply by disregarding the associated component d_j and its column in performing the penalty calculations. The main result for deriving the FP cuts based on the foregoing constructs can be stated as follows.

Theorem 1. Given a foundation function and penalty computations as defined in (5), the Foundation-Penalty (FP) cut for the case of a single binary variable x_k as given by

$$\sum_{j \in J} d_j x_j \geq P_{k1} x_k + P_{k0}(1 - x_k) \tag{6a}$$

and for the GUB-constrained case (4) as given by

$$\sum_{j \in J} d_j x_j \geq \sum_{k \in K} P_{k1} x_k \tag{6b}$$

yield valid inequalities for MIP. Moreover, in either case, under the condition (3) corresponding to an optimal basis for the LP relaxation $\overline{\text{MIP}}$ of MIP, if any of the penalties are positive for a currently fractional variable x_k in the LP solution, then (6) provides a *separating* inequality that deletes this LP solution.

Proof. The validity of (6a) follows directly from the penalty definitions (5) and the disjunction that { $x_k = 1$ or $x_k = 0$ }, and that for (6b) follows from (5a) and that $x_k = 1$ for exactly one $k \in K$, and is zero otherwise. Moreover, under the stated condition based on the LP relaxation $\overline{\text{MIP}}$, since the left-hand side of (6) is zero for this LP relaxation solution while the right-hand side is positive, the inequality (6) deletes this LP solution. This completes the proof. □

A direct extension of (6a) remains valid for any general integer restricted variable x_k as well, under the following conditions.

Corollary 1. Suppose that the foundation function yields

$$\Delta = \min\left\{\sum_{j\in J} d_j\, x_j :\ x \in \overline{X}\right\}, \tag{7a}$$

and that at optimality, some integer-restricted variable x_k takes on a value b_k that is fractional. Let P_k^+ and P_k^- be the respective values of the LP relaxations given by

$$P_k^+ = \text{minimum}\left\{\sum_{j\in J} d_j x_j : x \in \overline{X},\ x_k \geq \lfloor b_k \rfloor + 1\right\} \tag{7b}$$

$$P_k^- = \text{minimum}\left\{\sum_{j\in J} d_j x_j : x \in \overline{X}, x_k \leq \lfloor b_k \rfloor\right\}. \tag{7c}$$

Then the following inequality is valid

$$\sum_{j\in J} d_j x_j \geq P_k^+ (x_k - \lfloor b_k \rfloor) + P_k^- (\lfloor b_k \rfloor + 1 - x_k). \tag{8}$$

Moreover, if the foundation function conforms with (3) corresponding to the LP relaxation (2), then $\Delta = 0$ and furthermore, (8) is a separating inequality if either $P_k^+ > 0$ or $P_k^- > 0$.

Proof. Note that the disjunction $x_k \geq \lfloor b_k \rfloor + 1$ or $x_k \leq \lfloor b_k \rfloor$ is valid. In the former case, when $x_k = \lfloor b_k \rfloor + 1$, then clearly (8) is valid from (7b). Note that by (7a), for the problem

$$\nu(\theta) \equiv \min \left\{ \sum_{j \in J} d_j x_j : x \in \overline{X}, x_k \geq \theta \right\} \tag{9}$$

we have $\nu(\lfloor b_k \rfloor) = \Delta$. Since $\nu(\lfloor b_k \rfloor + 1) = P_k^+$ from (7b), and since $\nu(\theta)$ is a piecewise linear convex nondecreasing function of θ, and since $P_k^- \geq \Delta$, we have for all $x_k \geq \lfloor b_k \rfloor + 1$ with $x \in \overline{X}$, that

$$\sum_{j \in J} d_j x_j \geq \Delta + (P_k^+ - \Delta)(x_k - \lfloor b_k \rfloor) \geq \Delta + (P_k^+ - \Delta)(x_k - \lfloor b_k \rfloor)$$

$$+ (P_k^- - \Delta)(\lfloor b_k \rfloor + 1 - x_k) = P_k^+ (x_k - \lfloor b_k \rfloor) + P_k^- (\lfloor b_k \rfloor + 1 - x_k). \tag{10}$$

Hence, we have that (8) is true for this case. Likewise, by a parallel argument, (8) is true for all $x_k \leq \lfloor b_k \rfloor$, thereby establishing the validity of (8).

Moreover, if the foundation function conforms with (3), then $x_j = 0 \ \forall j \in J$ is optimal in (7a), yielding $\Delta = 0$. Furthermore, if $P_k^+ > 0$ or $P_k^- > 0$, then the right-hand side of (8) is positive when $x_k = b_k$, while the left-hand side of (8) is zero at the LP solution. Hence, (8) is a separating inequality in this case, and this completes the proof. □

Remark 1. The cut (8) (or (6) when based on similar LP relaxations), can be strengthened by attempting to reduce the coefficients d_j on the left-hand side, while preserving the validity of the penalties defining the right-hand sides. This can be achieved by examining the ranges of the d_j coefficients in (7a, b, c) that would leave the respective solutions and their values at optimality unaffected via a simple postoptimality analysis, and then revising d_j to the maximum of the lower interval end-points for these ranges for each $j \in J$. As such, one could even use a different foundation function for each of the children branching penalty computations (which may be called *sub-foundation functions*), and then compose an FP cut therefrom in a spirit similar to disjunctive cuts (see Section 5).

We now proceed to discuss the relationship of FP cuts with other classical cuts, which serves to provide additional insights into the selection of the basic elements defining FP cuts.

3. Relationship with Lifting Concepts

To illustrate this connection, consider the lifting of minimal cover inequalities for knapsack constraints as expounded by Crowder et al. (1983). Given a knapsack constraint of the general form

$$\sum_{j \in N_1} a_j x_j \geq b, \text{ where } 0 < a_j \leq b \ \forall j \in N_1, \text{ and where } N_1 \subseteq B, \tag{11}$$

let C be a minimal cover in the sense that

$$\sum_{j \in N_1 - C} a_j < b, \text{ but } \sum_{j \in N_1 - C} a_j + \min_{k \in C} \{a_k\} \geq b. \tag{12}$$

Hence,

$$\sum_{j\in C} x_j \geq 1 \tag{13}$$

is a valid (minimal cover) inequality. In order to lift (13) into the dimension of an additional variable x_t, $t \in N_1 - C$, and obtain a valid inequality

$$\sum_{j\in C} x_j \geq 1 + \alpha(1 - x_t), \tag{14}$$

Crowder et al. note that (14) is always valid when $x_t = 1$, while to maintain validity under the condition that $x_t = 0$ requires that

$$(1+\alpha) \leq \text{minimum}\left\{\sum_{j\in C} x_j : \sum_{j\in N_1} a_j x_j \geq b,\ x_j \text{ binary } \forall j \in N_1, \text{ and } x_t = 0\right\}. \tag{15}$$

Observe that this is akin to the derivation of an FP cut using $\sum_{j\in C} x_j$ as the foundation function, and considering the binary knapsack constraint as the set $x \in X$ (or its relaxation) in (5b). Accordingly, we can equate $P_{t0} = (1+\alpha)$. Furthermore, note that when we enforce $x_t = 1$ instead of $x_t = 0$ in the right-hand side of (15), we have from (12) that the objective value of the resulting knapsack problem is 1, so that we know *a priori* that $P_{t1} = 1.$ Hence, the FP cut (6a) in this case would be

$$\sum_{j\in C} x_j \geq P_{t1} x_t + P_{t0}(1 - x_t) = x_t + (1+\alpha)\,(1 - x_t) = 1 + \alpha\,(1 - x_t),$$

which coincides with (14). The analogy continues in a likewise fashion for subsequent lifting steps in this sequential process. In a similar vein, the FP cuts bear this conceptual relationship with the more general one- and zero-lifting described in Gu et al. [16], and to the simultaneous-GUB lifting described in Sherali and Lee [26], where the latter relates to (6b).

4. Higher-order FP Cuts

The FP cut (6a) has been derived with respect to the disjunction concerning a single variable x_k. The simple conceptual form of (6a) readily provides the flexibility of extending this cut to the simultaneous consideration of two or more variables, yielding higher-order FP cuts.

To illustrate, consider the case of a pair of binary variables x_k and x_ℓ. Analogous to (5), let

$$P_{k\ell}(p,q) \leq z_{k\ell}(p,q) \equiv \text{minimum}\left\{\sum_{j\in J} d_j x_j : x \in X, \text{ and } (x_k, x_\ell) = (p,q)\right\} \qquad (16a)$$

$$\text{for } (p,q) \in V \equiv \{(0,0),(1,0),(0,1),(1,1)\}. \qquad (16b)$$

Accordingly, analogous to (6a), we can assert that the following inequality is valid, because precisely one term on the right-hand side is nonzero, and equal to one, for any binary values of x_k and x_l, and then, the associated penalty is valid via (16).

$$\sum_{j\in J} d_j x_j \geq P_{k\ell}(1,1)x_k x_\ell + P_{k\ell}(1,0)x_k(1-x_\ell) + P_{k\ell}(0,1)(1-x_k)x_\ell + P_{k\ell}(0,0)(1-x_k)(1-x_\ell). \quad (17)$$

Observe that the right-hand side of (17) contains a quadratic product term $x_k x_\ell$. We can linearize this term by substituting $w_{k\ell} \equiv x_k x_\ell$ and accommodating bound-factor products as in Sherali and Adams [23], for example, to get

$$\sum_{j\in J} d_j x_j \geq \left[P_{k\ell}(1,1) + P_{kl}(0,0) - P_{k\ell}(1,0) - P_{k\ell}(0,1)\right] w_{k\ell}$$

$$+P_{k\ell}(1,0)x_k + P_{k\ell}(0,1)x_\ell + P_{k\ell}(0,0)(1-x_k-x_\ell) \tag{18a}$$

where

$$w_{k\ell} \le x_k, w_{k\ell} \le x_\ell, w_{k\ell} \ge 0, \text{ and } w_{k\ell} \ge x_k + x_\ell - 1. \tag{18b}$$

Notice that if the coefficient [•] of $w_{k\ell}$ in (18a) is positive, then by the nature of this inequality, it is only relevant to impose the last two constraints in (18b). Likewise, if this coefficient [•] is negative in (18a), only the first two inequalities in (18b) are relevant. Alternatively, we can use the relevant pair of inequalities from (18b) to project out $w_{k\ell}$ from (18a) and derive a pair of corresponding valid inequalities having only the original variables x_k and x_ℓ appearing in the right-hand side of (18a). Alternatively, the foregoing consideration of penalties for multiple variables can be used to formulate a disjunction that requires at least one of several inequalities to hold true, from which a disjunctive cut can be derived as explored more generally in the following section.

5. Relationship of FP Cuts to a Variety of Classical Valid Inequalities

The FP cuts bear a relationship with a variety of classical cuts such as disjunctive cuts, lift-and-project or RLT (reformulation-linearization technique) cuts, convexity cuts, mixed-integer rounding cuts, Gomory cuts, etc. As an illustration, to expose this association in the interesting context of GUB constraints, let us consider the MIP restrictions in the following form, where we have explicitly displayed a particular focal GUB constraint (4) based on a GUB set *K*, and where x_0 represents the vector of variables indexed by *N-K*, which are presently all relaxed to be continuous.

$$Ax_0 + \sum_{k \in K} A_k x_k \ge b \tag{19a}$$

$$\sum_{k \in K} x_k = 1 \tag{19b}$$

$$x_k \text{ binary } \forall\, k \in K, x_0 \geq 0. \tag{19c}$$

Note that while we have considered the constraints in (19a) to be all inequalities for ease in notation, equality constraints can also be included and handled in a similar fashion below. We can construct the convex hull representation of (19) by using the GUB special-structured RLT process described in Sherali et al. [25]. This involves multiplying (19a) and $x_0 \geq 0$ in (19c) by each x_k, $k \in K$, multiplying (19b) by x_0, applying the fact that $x_k^2 = x_k \;\; \forall\, k \in K$, and $x_k x_\ell = 0 \;\forall\, k, \ell \in K, k \neq \ell,$ and then substituting the vector y_k in place of the product term $x_0 x_k, \forall k \in K$. This yields the following representation

$$Ay_k - (b - A_k)x_k \geq 0 \quad \forall \;\; k \in \;K \tag{20a}$$

$$x_0 - \sum_{k \in K} y_k = 0 \tag{20b}$$

$$\sum_{k \in K} x_k = 1 \tag{20c}$$

$$x_k \geq 0 \;\forall k \in K, \; y_k \geq 0 \;\forall\, k \in K. \tag{20d}$$

Consequently, any valid inequality for (19) can be obtained (or is implied by) the projection of (20) onto the space of the original problem variables $(x_0, x_k \text{ for } k \in K)$. By LP duality (or Farkas' lemma), all such valid inequalities are obtained as weighted surrogates of (20) that zero out the coefficients for the new variables y_k, $\forall k \in K$. In other words, denoting $\pi_k \geq 0,\; \pi,$ and π_0 as the surrogate multipliers (dual variables) associated with (20a), (20b), and (20c), respectively, we obtain that any valid inequality is of the form

$$\pi^T x_0 + \sum_{k\in K} \left[\pi_0 - \pi_k^T \left(b - A_k\right)\right] x_k \geq \pi_0$$

where

$$\pi_k^T A - \pi^T \leq 0,\ \ \pi_k \geq 0, \forall k \in K.$$

Using the constraint $\sum_{k\in K} x_k = 1$, this yields any valid inequality in the form

$$\pi^T x_0 \geq \sum_{k\in K} \pi_k^T \left(b - A_k\right) x_k \tag{21a}$$

where

$$\pi_k^T A \leq \pi^T, \pi_k \geq 0\ \forall\, k \in K. \tag{21b}$$

Note that such cuts as obtained using the special-structured RLT process of Sherali et al. [25] are a generalization of the lift-and-project cuts of Balas et al. [4], where the latter are generated in a similar fashion but with respect to a single variable rather than with respect to a GUB set of variables as used above. Observe the relationship between (21a) and (6b). In particular, if we designate $\pi^T x_0$ as the foundation function, we can compute the corresponding penalty in (5a) as

$$P_{k1} = \text{ minimum } \left\{\pi^T x_0 :\ A x_0 \geq \left(b - A_k\right), x_0 \geq 0\right\}, \forall k \in K, \tag{22}$$

where the constraints of the problem in (22) correspond to fixing $x_k = 1$ in (19). Let π_k^* be an optimal dual multiplier associated with the (structural) constraints in (22), for each $k \in K$. Then we have,

$$P_{k1} = \pi_k^{*T} \left(b - A_k\right), \text{ where } \pi_k^{*T} A \leq \pi^T, \text{ and } \pi_k^* \geq 0, \forall k \in K. \tag{23}$$

Hence, the associated FP cut (6b) would then be given by

$$\pi^T x_0 \geq \sum_{k\in K} P_{k1} x_k = \sum_{k\in K} \pi_k^{*T} \left(b - A_k\right) x_k \tag{24}$$

which is precisely of the form (21), noting (23). In fact, given any π_k satisfying (21b), $\forall k \in K,$ the corresponding cut (21a) would be dominated by (24), since by duality in (22), $\pi_k^{*T}\left(b - A_k\right) \geq \pi_k^T\left(b - A_k\right)$ for all π_k feasible to (21b).

Cuts of the foregoing type can also be essentially viewed as *disjunctive cuts*. To expose this relationship further in a more general setting, consider the disjunction

$$\{\text{At least one of } A_k x \geq b_k,\ x \geq 0, \text{ must be satisfied, for some } k \in K\}. \tag{25}$$

The basic disjunctive principle of Balas [2, 3] and Jeroslow [18] (see also Glover [9] and Sherali and Shetty [29]), portends that any valid inequality for this disjunction can be derived as follows, by associating surrogate multipliers $\pi_k \geq 0$ with the constraints $A_k x \geq b_k, \forall k \in K$:

$$\pi^T x \geq \pi_0, \text{ where } \pi^T \geq \pi_k^T A_k \quad \forall k \in K, \quad \pi_0 \leq \pi_k^T b_k \quad \forall k \in K. \tag{26}$$

Observe that we can imagine that the disjunction (25) relates to a GUB constraint $\sum_{k \in K} y_k = 1,$ where for each $k \in K$, the binary variable y_k when put equal to 1 enforces the corresponding constraints $A_k x \geq b_k, x \geq 0$ to hold true. Accordingly, in the context of the FP cut, based on $\pi^T x$ as the foundation function, we can compute penalties via (5a) as

$$P_{k1} = \text{minimum } \left\{\pi^T x :\ A_k x \geq b_k, x \geq 0\right\} = \pi_k^{*T} b_k, \forall k \in K \tag{27}$$

where π_k^* is an optimal dual solution to (27), $\forall k \in K$. Hence, in particular, we have,

$$\pi_k^{*T} A_k \leq \pi^T, \quad \pi_k^* \geq 0, \quad \forall k \in K.$$

The corresponding FP cut (6b) would then be given as

$$\pi^T x \geq \sum_{k \in K} P_{k1} y_k. \tag{28}$$

This cut implies any valid cut (26) for the given foundation function $\pi^T x$, because from (27), noting by duality that

$$\pi_k^* b_k \geq \pi_k^T b_k \quad \forall \quad \pi_k \geq 0 \text{ such that } \pi_k^T A_k \leq \pi^T, \tag{29}$$

we have,

$$\pi^T x \geq \sum_{k \in K} P_{k1} y_k = \sum_{k \in K} \left(\pi_k^{*T} b_k\right) y_k \geq \sum_{k \in K} \left(\pi_k^T b_k\right) y_k \geq \sum_{k \in K} \pi_0 y_k = \pi_0 . \tag{30}$$

Observe that (30) prompts an alternative form of the FP cut (28) that can be imposed without involving the *y*-variables. Assuming $P_{k1} > 0 \ \forall k \in K$ to illustrate, (28) can be posed as the disjunction that at least one of $(1/P_{k1})\pi^T x \geq 1,\ k \in K$, must hold true, leading to the cut

$$\max_{k \in K} \{(1/P_{k1})\pi\}^T x \geq 1, \tag{31}$$

where the max operation is performed componentwise. This form (31) can be viewed as a *balanced FP cut*. Moreover, multiple applications of such disjunctive formulations and scalings could be conducted to derive suitable cuts that penetrate deeper in desired dimensions.

This general relationship with disjunctive cuts extends the relationship of FP cuts to the convexity cuts of Glover [8] when based on polyhedral convex sets, to Gomory's (mixed) integer cuts [13], and to mixed-integer rounding cuts discussed in Marchand and Wolsey [19] (which are in essence Gomory cuts), all of which are derived based on the formulation of specific disjunctions of the type (25). In each of these cases, having obtained some foundation function via suitable surrogates of the underlying disjunctive sets as in (21) or (26), the resulting disjunctive cut can be further tightened through the FP cut viewpoint by deriving tighter penalties in (22) or (27). This could be accomplished by imposing further valid restrictions (e.g., integrality on some presently relaxed variables) in the penalty problems (22) or (27), leading to stronger versions of the cuts (24) or (28). This ability to produce deeper FP cuts by generating stronger penalties is a particularly useful feature. (For example, the mixed-integer rounding cuts and the Gomory cuts are generated by Corollary 1 for the simple penalty value that was the first to be used in MIP methods.) This rich association with other cutting plane proposals, allowing them to be derived and analyzed by reference to the FP representation, provides insights into the flexibility and latent capability inherent in this class of FP cuts.

6. Concluding Remarks

The Foundation-Penalty (FP) cuts offer a previously unavailable opportunity to exploit penalty calculations of the type customarily used in branch-and-bound, thereby yielding a new utility for these calculations that supplements their role in fathoming nodes and in selecting branches of the branch-and-bound tree. Consequently, the FP cuts are particularly relevant for use in branch-and-cut methods.

The introduction of this new class of cutting planes also opens up several areas of related research. The latitude to select the foundation function in order to bias the cut to extend more deeply in particular dimensions invites an investigation of alternative strategies for generating these functions. Similarly, the trade-offs involved in employing more advanced penalty calculations in the process of generating the cutting planes warrant investigation. In particular, higher-order penalties may provide a different degree of advantage for FP cuts than for branch-and-bound fathoming operations, since the latter are only relevant in the case where it is possible to determine infeasibility or to establish that the objective function exceeds an admissible bound.

The determination of which GUB sets from a given collection provide the best source for FP cuts also invites investigation. Similarly, the selection of single or multiple variables in generating first-order or higher-order FP cuts is an interesting avenue for further research. We anticipate that MIP problems in which GUB constraints are numerous and include a large portion of the integer variables are likely to provide the most useful applications for these cutting planes. The fact that the FP cuts in such settings are based on selecting GUB sets rather than individual variables as a foundation for creating the cutting plane structure imparts them a novel property whose consequences likewise deserve study.

References

[1] D. Applegate, R. Bixby, W. Cook and V. Chvatal (1996), Personal Communication in J. T. Linderoth and M. W. P. Savelsbergh, "A Computational Study of Search Strategies for Mixed Integer Programming," *INFORMS Journal on Computing,* Vol. 11, pp. 173-187, 1999.

[2] E. Balas, (1979), "Disjunctive Programming," *Annals of Discrete Mathematics*, 5, pp. 3-51.

[3] E. Balas, (1998), "Disjunctive Programming: Properties of the Convex Hull of Feasible Points," *Discrete Applied Mathematics,* Vol. 89, Nos. 1-2, pp. 3-44.

[4] E. Balas, S. Ceria and G. Cornuejols (1993), "A Lift-and-Project Cutting Plane Algorithm for Mixed 0-1 Programs," *Mathematical Programming,* Vol. 58, pp. 295-324.

[5] E. Balas and E. Zemel (1975), "Facets of the Knapsack Polytope from Minimal Covers," *SIAM Journal on Applied Mathematics,* Vol. 34, pp. 119-148.

[6] H. Crowder, E. L. Johnson and M. W. Padberg (1983), "Solving Large-Scale Zero-One Linear Programming Problems," *Operations Research,* vol. 31, pp. 803-834.

[7] F. Glover, (1970) "Faces of the Gomory Polyhedron, "*Integer and Nonlinear Programming II*, Jean Abadie, editor, North-Holland Publishing Company.

[8] F. Glover, (1973) "Convexity Cuts for Multiple Choice Problems," *Discrete Mathematics*, Vol. 3, No. 1, pp. 86-100.

[9] F. Glover, (1975) "Polyhedral Annexation in Mixed Integer and Combinatorial Programming," *Mathematical Programming*, Vol. 8, pp. 161-188.

[10] F. Glover, H. D. Sherali, and Y. Lee (1997) "Generating Cuts from Surrogate Constraint Analysis for Zero-One and Multiple Choice Programming," *Computational Optimization and Applications,* Vol. 8, No. 2, pp. 151-172.

[11] F. Glover and D. Sommer (1976) "Inequalities for Mixed Integer Programs with Structure," *Naval Research Logistics Quarterly,* Vol. 23, No. 4, pp. 603-609.

[12] R.E. Gomory, (1960a) "Solving Linear Programming Problems in Integers," R.E. Bellman and M. Hall, Jr., eds. *Combinatorial Analysis,* American Mathematical Society, pp. 211-216.

[13] R.E. Gomory, (1960b) "An Algorithm for the Mixed Integer Problem," Research Memorandum RM-2597, Rand Corporation, Santa Monica.

[14] R.E. Gomory, (1969), "Some Polyhedra Related to Combinatorial Problems," *Linear Algebra and Its Applications,* Vol. 2, pp. 451-558.

[15] Z. Gu, G. L. Nemhauser, and M. W. P. Savelsbergh (1995), "Sequence Independent Lifting," working paper, Georgia Institute of Technology, Atlanta, GA.

[16] Z. Gu, G. L. Nemhauser, and M. W. P. Savelsbergh (1999), "Lifted Flow Covers for Mixed 0-1 Integer Programs," *Mathematical Programming,* Vol. 85, pp. 439-468.

[17] P.L. Hammer, E. L. Johnson, and U. N. Peled (1975) "Facets of Regular 0-1 Polytopes," *Mathematical Programming,* Vol. 8, pp. 179-206.

[18] R.G. Jeroslow, (1977), "Cutting Plane Theory: Disjunctive Methods," *Annals of Discrete Mathematics,* Vol. 1, pp. 293-330.

[19] H. Marchand and L. A. Wolsey (2001) "Aggregation and Mixed Integer Rounding to Solve MIPS," *Operations Research,* Vol. 49, No. 3, pp. 363-371.

[20] G.L. Nemhauser and L.A. Wolsey (1988), "*Integer and Combinatorial Optimization,"* John Wiley & Sons, Inc., New York, N.Y.

[21] M.W. Padberg, (1973), "On the Facial Structure of Set Packing Polyhedra," *Mathematical Programming,* Vol. 5, pp. 199-215.

[22] R. Rardin and M. H. Karwan (1984), "Surrogate Dual Multiplier Search Procedures in Integer Programming," *Operations Research,* Vol. 32, pp. 52-69.

[23] H.D. Sherali and W. P. Adams, (1990), "A Hierarchy of Relaxations Between the Continuous and Convex Hull Representations for Zero-One Programming Problems," *SIAM Journal on Discrete Mathematics,* Vol. 3, No. 3, pp. 411-430.

[24] H.D. Sherali and W. P. Adams, (1994), "A Hierarchy of Relaxations and Convex Hull Characterizations for Mixed-Integer Zero-One Programming Problems," *Discrete Applied Mathematics*, Vol. 52, pp. 83-106.

[25] H.D. Sherali, W. P. Adams, and P. J. Driscoll, (1998), "Exploiting Special Structures in Constructing a Hierarchy of Relaxations for 0-1 Mixed Integer Problems," *Operations Research*, Vol. 46, No. 3, pp. 396-405.

[26] H.D. Sherali. and Y. Lee (1995) "Sequential and Simultaneous Liftings of Minimal Cover Inequalities for Generalized Upper Bound Constrained Knapsack Polytopes," *SIAM Journal on Discrete Mathematics,* Vol. 8, No. 1, pp. 133-153.

[27] H.D. Sherali, Y. Lee, and W. P. Adams (1995) "A Simultaneous Lifting Strategy for Identifying New Classes of Facets for the Boolean Quadric Polytope," *Operations Research Letters,* Vol. 17, No. 1, pp. 19-26.

[28] H.D. Sherali, Y. Lee, and Y. Kim (2001) "Partial Convexification Cuts for 0-1 Mixed-Integer Programs," Working paper, Virginia Polytechnic and State University, Blacksburg, VA.

[29] H.D. Sherali and C. M. Shetty (1980), *Optimization with Disjunctive Constraints*, Series in Economics and Mathematical Systems, Vol. 181, Springer-Verlag, Berlin-Heidelberg-New York.