Some Classes of Valid Inequalities and Convex Hull Characterizations for Dynamic Fixed-charge Problems under Nested Constraints

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Abstract

This paper studies the polyhedral structure of dynamic fixed-charge problems that have nested relationships constraining the flow or activity variables. Constraints of this type might typically arise in hierarchical or multi-period models and capacitated lot-sizing problems, but might also be induced among choices of key variables via an LP-based post-optimality analysis. We characterize several classes of valid inequalities and inductively derive convex hull representations in a higher dimensional space using lifting constructs based on the Reformulation-Linearization Technique. Relationships with certain known classes of valid inequalities for single item capacitated lot-sizing problems are also identified.

Keywords: Dynamic fixed-charge problems, capacitated lot-sizing, Reformulation-linearization Technique, valid inequalities, convex hull.

Fixed-charge problems, notably including network flow and facility location fixed-charge problems, occupy a central place among classical mixed-integer programming models. An extensive literature of practical applications and of proposed solution procedures has emerged, attesting to the importance and challenge of this class of problems. Applications include natural gas pipeline systems (Rothfarb et al., 1970), offshore platform drilling (Balas and Padberg, 1976), bank account location (Cornuejols et al., 1977), distribution center location (Nozick and Turnquist, 1998a, 1998b), telecommunication network switching (Luna et al., 1987), and network design (Mirzain, 1985; Crainic et al., 2001). Several other network-related applications are also discussed in Glover et al., 1992.

Solution methods for various types of fixed-charge problems have ranged across the spectrum of approaches spanning Lagrangian relaxation with branch-and-bound (Cruz et al., 1998), Lagrangian relaxation with heuristics (Hochbaum and Segev, 1989), bundle-based relaxations (Crainic et al., 2001), branch-and-bound with Benders decomposition (Magnanti et al., 1986), branch-and-bound with cutting planes (Cabot and Erenguc, 1984; Suhl, 1985; Padberg et al., 1985), tabu search (Sun et al., 1998) and iterated scaling (Glover, 1994; Kim and Pardalos, 1999). For the setting of network problems, specialized cutting planes have also been proposed (Barahona, 1986; Bienstock and Günlük, 1996; Bienstock and Muratore, 1997; Stallaert, 2000).

In this paper, we address the issue of generating cutting planes for dynamic fixed-charge problems without restriction to network flow models, but where the feasible region is constrained by inequalities exhibiting a certain nesting property that typically arise in hierarchical or multi-period decision process models (hence, the term *dynamic*).

Accordingly, let us consider the following mixed-integer 0-1 region, X_n, defined in terms of some n continuous variables $x \in R^n$ along with an associated set of n binary variables $y \in B^n$, where each x_j is bounded on $[0, \alpha_j]$ if $y_j = 1$, and is zero otherwise, and where the flow or activity levels $x_1, \ldots, x_n$ satisfy a nested set of generalized upper bounding (GUB) constraints as stated below.

$X_n = \{(x, y) \in R_+^n \times B^n :$

$$0 \leq x_j \leq \alpha_j y_j, \ \forall j=1, \ldots, n \tag{1a}$$

$$\sum_{j=1}^{k} x_j \leq \beta_k, \ \forall k=2, \ldots, n \tag{1b}$$

$$y \text{ binary}\}, \tag{1c}$$

where we assume that $\alpha_j > 0$, $\forall j=1, \ldots, n$, and that

$$\max\{\alpha_1, \alpha_2\} \leq \beta_2 < \alpha_1 + \alpha_2, \text{ and } \max\{\alpha_k, \beta_{k-1}\} \leq \beta_k < \alpha_k + \beta_{k-1}, \ \forall k=3, \ldots, n. \tag{2}$$

Observe that Assumption (2) simply obviates possible coefficient reductions and elimination of redundant constraints. For example, if either α_1 or α_2 is greater than β_2, then noting (1a) and that $x_1 + x_2 \leq \beta_2$ from (1b), we could reduce such an α–coefficient to β_2. Likewise, if $\beta_2 \geq \alpha_1 + \alpha_2$, then $x_1 + x_2 \leq \beta_2$ is implied by (1a), and would then be redundant. Similarly, if either α_k or β_{k-1} exceeds β_k, then it can be legitimately reduced to β_k, and if $\beta_k \geq \alpha_k + \beta_{k-1}$, then (1b) for k is implied by (1b) for (k-1) along with $x_k \leq \alpha_k$ from (1a).

The constraints defining X_n might typically be a subset of the restrictions that model some dynamic fixed-charge problem that exhibits such a nested structure. Alternatively, this nested inequality structure could be generated for some key subset of variables as desired via a suitable LP post-optimization, if it is not otherwise already explicitly present. This could be done by successively maximizing the closely-related expressions on the left-hand-side of (1b) for k=2,…,n. The set X_n also arises in the context of single item capacitated lot-sizing problems as demonstrated by Atamturk and Munoz (2003). In this context, considering the demand d_t for some product over periods t=1,…,n, and letting w_t denote the production or order quantity during period t, i_t denote the available inventory at the beginning of period t (or at the end of period t-1), and letting c_t denote the production capacity during period t, we can model this multi-period production-inventory lot-sizing scenario as follows:

$$i_t + w_t = i_{t+1} + d_t, \quad \forall\ t=1,\ldots,n$$

$$0 \le w_t \le c_t\, z_t,\ \forall\ t=1,\ldots,n$$

$$z_t \in \{0,1\},\ \forall\ t=1,\ldots,n \text{ and } i_{n+1} \equiv 0.$$

Here, whenever a production run is made during period t (i.e., $w_t > 0$), then the binary variable z_t necessarily takes on a value of one, and would correspondingly incur some fixed-charge cost. Now, consider the transformation

$$x_j = w_{n-j+1},\ \forall\ j=1,\ldots,n, \text{ and } y_j = z_{n-j+1},\ \forall\ j=1,\ldots,n \tag{3a}$$

and set

$$\alpha_j = c_{n-j+1},\ \forall\ j=1,\ldots,n, \text{ and } \beta_j = \sum_{k=1}^{j} d_{n-k+1},\ \forall\ j=1,\ldots,n. \tag{3b}$$

Then, eliminating the inventory variables i_t, for t=1,…,n by considering the above production-inventory balance constraints in the reverse order for t=n,…,1, produces the following equivalent set of constraints for the above lot-sizing polytope, where the slack in the first set of constraints is given by the inventory variable i_{n-k+1}, for each k=1,…,n.

$$\sum_{j=1}^{k} x_j \le \beta_k,\ \forall\ k=1,\ldots,n \tag{3c}$$

$$0 \le x_j \le \alpha_j\, y_j,\ \forall\ j=1,\ldots,n \tag{3d}$$

$$y_j \in \{0,1\},\ \forall\ j=1,\ldots,n. \tag{3e}$$

Observe that if we take $\alpha_1 = \beta_1$, then (3c) – (3e) is precisely the set X_n described by (1). We note here that it is also usually assumed that the initial inventory i_1 at the beginning of period t=1 is known and, without loss of generality, taken to be zero, so that (3c) for k=n becomes

$$\sum_{j=1}^{n} x_j = \beta_n . \tag{3f}$$

Barany et al. (1984a) have considered the uncapacitated version of (3c) – (3f) in which $\alpha_j \equiv \beta_n$, $\forall\ j=1,\ldots,n$, and have provided a complete convex hull description for this polytope. Pochet (1988) has extended this work to derive a family of valid inequalities for the capacitated version (3c) – (3f), focusing mainly on the equal capacity case for which he demonstrates that a large subclass of these inequalities is facet-defining. Loparic et al. (2003) have examined dynamic knapsack polytopes as multi-dimensional knapsack sets having an additional continuous variable, and have explored relationships of such sets with (relaxations of) discrete and continuous single item capacitated lot-sizing problems in order to derive strong valid inequalities for the latter problems. Atamturk and Munoz (2003) have introduced a new class of so-called bottleneck cover valid inequalities for (3c) – (3e) that are shown to delete all fractional vertices of the corresponding continuous linear programming relaxation. They have also studied various liftings and facet-inducing properties of this class of valid inequalities. As a further extension to (3c) – (3e), Atamturk and Kucukyavuz (2003) have additionally imposed either constant or fixed-charge-based bounds on the inventory variables (slacks in (3c)), and have studied the polyhedral structure of the resulting set, describing various facet-defining inequalities along with separation routines. We also refer the interested reader to the paper by Van Vyve and Ortega (2003) for related convex hull results, and to the survey by Pochet and Wolsey (1995) for a further discussion on the literature pertaining to lot-sizing problems.

In what follows, we will characterize certain valid inequalities and higher dimensional convex hull representations for X_n, in order to tighten the relaxation of this underlying parent problem. Some of these classes of valid inequalities are related to certain known inequalities for the lot-sizing polytope, while others are new, as discussed in the sequel. We remark here that if the constraints (1b) have some general positive coefficients a_j for each x_j, j=1, …, n, in the form

$$\sum_{j=1}^{k} a_j x_j \leq \beta_k, \; \forall\, k=2, \ldots, n,$$

then we can simply scale the problem to transform it into the form of X_n by defining variables $x'_j = a_j x_j$, j=1, …, n. For such a transformed or scaled region, given that (2) is satisfied, all the results derived herein would continue to hold true.

We begin in the next section by deriving a class of nested valid inequalities for X_n. We provide some insights into deriving these inequalities via either an application of the Reformulation-Linearization Technique (RLT) of Sherali and Adams (1990, 1994), or via a specific related lifting process. Following this, we show in Section 2 that for the case of $n = 2$, this produces the convex hull of X_2. However, we demonstrate that this is not the case when $n \geq 3$, and this illustration leads to additional classes of valid inequalities for X_n in Section 3, for $n \geq 3$. We also discuss relationships with certain known classes of valid inequalities for the lot-sizing polytope. Finally, we close in Section 4 by developing an inductive scheme for constructing the convex hull representation for X_n in a higher dimensional space.

1. A Class of Nested Valid Inequalities

Let us begin by considering the case of $n = 2$ as addressed in Proposition 1 below. Note that this case has no nested structure, and so, the corresponding valid inequality described in Proposition 1 is precisely the special case $(S, \varnothing)$ of the (S, L) flow cover inequality defined by

Proposition 3 of Padberg et al. (1985) for arbitrary n. Nonetheless, we provide a proof to demonstrate an insightful alternative derivation process, which will then lead to an inductive scheme for deriving a new prescribed class of valid inequalities in closed-form for $n \geq 3$.

Proposition 1. For $n = 2$, the following is a valid inequality for X_2:

$$x_1 + x_2 \leq (\beta_2 - \alpha_2)y_1 + (\beta_2 - \alpha_1)y_2 + (\alpha_1 + \alpha_2 - \beta_2). \tag{4}$$

Proof. Adopting the RLT process, let us define y_{12} as the linearization of the product term y_1y_2, and note that

$$y_{12} \geq y_1 + y_2 - 1 \tag{5}$$

for any binary values of y_1 and y_2. Now, consider the surrogate formed by multiplying the constraints from (1a) and (1b) by the nonnegative factors y_{12} and $(1-y_{12})$ as shown below, and summing these inequalities (where $\oplus$ denotes this surrogation or summing process):

$$[x_1 \leq \alpha_1 y_1](1-y_{12}) \oplus [x_2 \leq \alpha_2 y_2](1-y_{12}) \oplus [x_1 + x_2 \leq \beta_2]y_{12}. \tag{6}$$

Upon using the fact that $y_1y_{12} = y_2y_{12} = y_{12}$, we get

$$x_1 + x_2 \leq \alpha_1 y_1 + \alpha_2 y_2 - y_{12}\,(\alpha_1 + \alpha_2 - \beta_2). \tag{7}$$

Noting that $(\alpha_1 + \alpha_2 - \beta_2) > 0$ from (2), and using $-y_{12} \leq -y_1 - y_2 + 1$ from (5) within (7), we get (4). This completes the proof. □

The following result inductively generates a nested class of valid inequalities of type (4) for $n \geq 3$. For notational convenience, we will henceforth adopt the RLT terminology whereby $[\bullet]_L$ represents the linearization of $[\bullet]$ under the RLT substitution of a single variable for each specific product term. For example, in particular, $y_{12} \equiv [y_1\, y_2]_L$. Furthermore, let us denote

$$J_k = \{1, \ldots, k\}, \text{ and let } y_{J_k} = \left[\prod_{j=1}^{k} y_j\right]_L. \tag{8}$$

Observe that we have the following readily verified relationship holding true:

$$y_{J_k} \geq \sum_{j=1}^{k} y_j - (k-1), \ \forall \ k=2, \ldots, n. \tag{9}$$

Proposition 2. The following class of nested inequalities are valid for X_n for each k=2, …, n:

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} \pi_j^k y_j + \pi_0^k \tag{10a}$$

where for each k=3, …, n, we have

$$\pi_j^k = \pi_j^{k-1} - (\beta_{k-1} + \alpha_k - \beta_k), \ \forall \ j=1, \ldots, k\text{-}1 \tag{10b}$$

$$\pi_k^k = (\beta_k - \beta_{k-1}) \tag{10c}$$

and

$$\pi_0^k = \pi_0^{k-1} + (k-1)(\beta_{k-1} + \alpha_k - \beta_k) \tag{10d}$$

and where for $k = 2$, we have

$$\pi_1^2 = (\beta_2 - \alpha_2), \ \pi_2^2 = (\beta_2 - \alpha_1), \text{ and } \pi_0^2 = (\alpha_1 + \alpha_2 - \beta_2). \tag{10e}$$

In particular, we have the sum of the valid inequality coefficients yielding

$$\sum_{j=1}^{k} \pi_j^k + \pi_0^k = \beta_k, \ \forall \ k=2, \ldots, n. \tag{11}$$

Proof. We establish this result by induction on k. For $k = 2$, the inequality given by (10a, e) is valid from (4) of Proposition 1. Moreover, noting (10e), we have that (11) holds true.

Hence, suppose that the result is true for some k-1, and consider the case for k, where $k \in \{3, \ldots, n\}$. Using (10a) for the case of k-1, and (1a) and (1b) for the case of k, consider the following RLT product constraint surrogate as in the proof of Proposition 1, where y_{J_k} is defined by (8).

$$\left[\sum_{j=1}^{k-1} x_j \le \sum_{j=1}^{k-1} \pi_j^{k-1} y_j + \pi_0^{k-1}\right](1-y_{J_k}) \oplus [x_k \le \alpha_k y_k](1-y_{J_k}) \oplus \left[\sum_{j=1}^{k} x_j \le \beta_k\right](y_{J_k}) . \qquad (12)$$

Using the fact that $y_j y_{J_k} \equiv y_{J_k}, \ \forall \ j \in J_k \equiv \{1, ..., k\}$, we get

$$\sum_{j=1}^{k} x_j \le \sum_{j=1}^{k-1} \pi_j^{k-1} y_j + \alpha_k y_k + \pi_0^{k-1} - y_{J_k}\left[\sum_{j=1}^{k-1} \pi_j^{k-1} + \pi_0^{k-1} + \alpha_k - \beta_k\right]. \qquad (13)$$

By the induction hypothesis on (11), the term [•] in (13) is equal to [$\beta_{k\text{-}1} + \alpha_k - \beta_k$], which is positive by (2). Consequently, applying the inequality (9) in (13), we get

$$\sum_{j=1}^{k} x_j \le \sum_{j=1}^{k-1} \pi_j^{k-1} y_j + \alpha_k y_k + \pi_0^{k-1} - \left[\sum_{j=1}^{k} y_j - (k-1)\right](\beta_{k-1} + \alpha_k - \beta_k),$$

which is precisely of the form (10). Moreover, from (10b, c, d) and the induction hypothesis on (11) for the case of k-1, we obtain

$$\sum_{j=1}^{k} \pi_j^k + \pi_0^k = \left[\sum_{j=1}^{k-1} \pi_j^{k-1} + \pi_0^{k-1}\right] + (\beta_k - \beta_{k-1}) = \beta_k ,$$

or that (11) continues to hold true for the case of k. This completes the proof. □

Remark 1 (Derivation via a Lifting Argument):

The inequalities (4), in particular, and (10) in general, can also be derived via a lifting argument. To illustrate, consider the inequality (4). Note that from (1a), we have the following valid inequality:

$$(x_1+x_2) \le \alpha_1 y_1 + \alpha_2 y_2. \qquad (14)$$

We can lift this in the dimension of the product variable y_{12} as follows, using a coefficient $\alpha \ge 0$ for y_{12}:

$$(x_1+x_2) \le \alpha_1 y_1 + \alpha_2 y_2 - \alpha y_{12}. \qquad (15)$$

From (14), we have that (15) remains valid whenever $y_{12} = 0$, i.e., y_1 or y_2 equals zero. To maintain validity of (15) in the remaining case of $y_1 = y_2 = 1$, whenever $y_{12} = 1$, we must have

$$\alpha \leq \alpha_1 + \alpha_2 - \max \{(x_1 + x_2): (x, y) \in X_2 \text{ with } y_1 = y_2 = 1\}. \tag{16}$$

By (1b) and (2), the maximum value in (16) is given by β_2, by which we can take $\alpha = (\alpha_1 + \alpha_2 - \beta_2)$, whereby (15) leads to the valid inequality (7). This in turn yields the desired inequality (4) upon using (5) as in the proof of Proposition 1.

Similarly, we can derive (10), in general, via such a lifting process. This can be accomplished by inductively starting with the valid inequality (10a) for the case of k-1, for some $k \geq 3$, along with (1a) for the case k, to get

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k-1} \pi_j^{k-1} y_j + \pi_0^{k-1} + \alpha_k y_k \,. \tag{17}$$

Lifting this with a coefficient $-\alpha y_{J_k}$ on the right-hand side, we can derive α as in (16) under the relevant condition $y_1 = \ldots = y_k = 1$ via

$$\alpha \leq \sum_{j=1}^{k-1} \pi_j^{k-1} + \pi_0^{k-1} + \alpha_k - \max\left[\sum_{j=1}^{k} x_j : (x, y) \in X_k \text{ with } y_1 = \ldots = y_k = 1\right].$$

Hence, noting (1b) and (11), we can take

$$\alpha = \sum_{j=1}^{k-1} \pi_j^{k-1} + \pi_0^{k-1} + \alpha_k - \beta_k = (\beta_{k-1} + \alpha_k - \beta_k)\,, \tag{18}$$

which leads to (13), and thereby to (10) for the case of k as in the proof of Proposition 2. □

Example 1

To illustrate Propositions 1 and 2, consider the case of $n = 3$, with X_3 being described as follows.

$$0 \leq x_1 \leq 6y_1 \tag{19a}$$

$$0 \le x_2 \le 7y_2 \tag{19b}$$

$$0 \le x_3 \le 8y_3 \tag{19c}$$

$$x_1 + x_2 \le 10 \tag{19d}$$

$$x_1 + x_2 + x_3 \le 11 \tag{19e}$$

$$(y_1, y_2, y_3) \text{ binary.} \tag{19f}$$

Hence, we have $\alpha_1 = 6$, $\alpha_2 = 7$, $\alpha_3 = 8$, $\beta_2 = 10$, and $\beta_3 = 11$, with (2) holding true. Applying Proposition 1 for the case of $n = 2$, we have that the inequality (4) is given by

$$x_1 + x_2 \le 3y_1 + 4y_2 + 3. \tag{20}$$

Next, inductively applying Proposition 2 for the case of $k = 3$, we get from (10b, c, d) using $(\beta_{k-1} + \alpha_k - \beta_k) = (10 + 8 - 11) = 7$, and $(\beta_k - \beta_{k-1}) = 11 - 10 = 1$, that $\pi_1^3 = 3 - 7 = -4$, $\pi_2^3 = 4 - 7 = -3$, $\pi_3^3 = 1$, and $\pi_0^3 = 3 + (2)(7) = 17$. This leads to (10a) as given by

$$x_1 + x_2 + x_3 \le -4y_1 - 3y_2 + y_3 + 17. \tag{21}$$

We mention here that not only is (20) facet-defining for conv(X_2), but also, as shown in general in the next section, it serves to completely describe conv(X_2). On the other hand, as we show later in Example 2, the inequality (21) is dominated by the facet-defining inequality $x_1 + x_2 + x_3 \le 10 + y_3$. Observe that as shown in Remark 1, (21) is essentially derived by lifting the facet (20) for conv(X_2) combined with (19c) according to $x_1 + x_2 + x_3 \le 3y_1 + 4y_2 + 8y_3 + 3 - \alpha y_{J_3}$, where $\alpha = 7$ in this case. Evidently, using the projection of this onto the original variable space via the inequality $y_{J_3} \ge y_1 + y_2 + y_3 - 2$, which yields (21), fails to preserve the facet-inducing property in this inductive process. Nonetheless, we describe later in Section 4 an inductive process for generating conv(X_n) in a higher dimensional representation.

Remark 2. Note that in lieu of following the inductive scheme of Proposition 2 for $k = 3$, if we had directly adopted the strategy of Proposition 1 that was used for $k = 2$, we would have derived a weaker cut than (21) (this is generally true). To illustrate, note that such a direct derivation would have used the RLT construct

$$(x_1 \leq \alpha_1 y_1)(1 - y_{123}) \oplus (x_2 \leq \alpha_2 y_2)(1 - y_{123}) \oplus (x_3 \leq \alpha_3 y_3)(1 - y_{123}) \oplus (x_1 + x_2 + x_3 \leq \beta_3)\, y_{123}$$

leading to the cut

$$(x_1 + x_2 + x_3) \leq \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 - y_{123}\,(\alpha_1 + \alpha_2 + \alpha_3 - \beta_3).$$

Using $-y_{123} \leq -y_1 - y_2 - y_3 + 2$ from (9) for $J_k = \{1, 2, 3\}$, this yields

$$(x_1 + x_2 + x_3) \leq -(\alpha_2 + \alpha_3 - \beta_3)y_1 - (\alpha_1 + \alpha_3 - \beta_3)y_2 - (\alpha_1 + \alpha_2 - \beta_3)y_3 + 2(\alpha_1 + \alpha_2 + \alpha_3 - \beta_3). \quad (22)$$

On the other hand, using (4) and (10) for the case $k = 3$, Proposition 2 yields the following valid inequality for this case:

$$(x_1 + x_2 + x_3) \leq -(\alpha_2 + \alpha_3 - \beta_3)y_1 - (\alpha_1 + \alpha_3 - \beta_3)y_2 + (\beta_3 - \beta_2)y_3 + (\alpha_1 + \alpha_2 + 2\alpha_3 + \beta_2 - 2\beta_3). \quad (23)$$

Observe that (23) implies (22) in general, because its right-hand-side is generally smaller than that of (22), as seen by noting that the former minus the latter is given by

$$(\alpha_1 + \alpha_2 - \beta_2)y_3 - (\alpha_1 + \alpha_2 - \beta_2) = -(\alpha_1 + \alpha_2 - \beta_2)\,(1 - y_3) \leq 0$$

for any $y_3 \leq 1$, noting that $\alpha_1 + \alpha_2 - \beta_2 > 0$ by (2). For Example 1 above, (22) is given by

$$x_1 + x_2 + x_3 \leq -4y_1 - 3y_2 - 2y_3 + 20, \quad (24)$$

while the inequality (23) is given by (21), where the right-hand-side of (21) minus that of (24) equals $-3(1 - y_3) \leq 0$. □

2. Convex Hull Characterization for $n = 2$

Let the set X_2 be defined as in (1), restated explicitly below for the sake of convenience:

$$X_2 = \{(x_1, x_2, y_1, y_2):\ 0 \leq x_j \leq \alpha_j y_j \ \text{ for } \ j=1, 2,\ x_1 + x_2 \leq \beta_2,\ (y_1, y_2) \text{ binary}\}. \quad (25)$$

Consider the set Z_2 defined as follows, by incorporating the valid inequality (4) into X_2 and relaxing the binary restrictions.

$$Z_2 = \{(x_1, x_2, y_1, y_2):\ 0 \le x_j \le \alpha_j y_j \ \text{ for } \ j=1, 2, \tag{26a}$$

$$x_1 + x_2 \le (\beta_2 - \alpha_2)y_1 + (\beta_2 - \alpha_1)y_2 + (\alpha_1 + \alpha_2 - \beta_2) \tag{26b}$$

$$0 \le y_1 \le 1,\ \ 0 \le y_2 \le 1\}. \tag{26c}$$

Observe that we have dropped $x_1 + x_2 \le \beta_2$ in Z_2 since this is implied by (26b), because noting (2), we have that the right-hand-side in (26b) for any $0 \le y_j \le 1$, $\forall\ j$, satisfies $(\beta_2 - \alpha_2)y_1 + (\beta_2 - \alpha_1)y_2 + (\alpha_1 + \alpha_2 - \beta_2) \le (\beta_2 - \alpha_2) + (\beta_2 - \alpha_1) + (\alpha_1 + \alpha_2 - \beta_2) = \beta_2$. Indeed, as established by the next result, Z_2 characterizes conv(X_2), where conv(•) denotes the convex hull operation. (The convex hull of the uncapacitated version of X_n under (3f) and with $\alpha_j = \beta_n$, $\forall\ j=1,\ldots,n$, is described in Barany et al. (1984). To our knowledge, the following result is new.)

Proposition 3. conv(X_2) = Z_2.

Proof. Since (26b) is valid for X_2 by Proposition 1, we have that conv(X_2) $\subseteq Z_2$. Hence, it is sufficient to show that all vertices of Z_2 (denoted vert(Z_2)) are feasible to X_2. In particular, noting that (26b, c) implies $x_1 + x_2 \le \beta_2$ in (25), it is sufficient to show that y is binary valued at all points in vert(Z_2). Observe that for any vertex of Z_2 at which (26b) is inactive, by the separable structure of (26a) and (26c) over the (x_1, y_1) and (x_2, y_2) spaces, we see that this claim is true. Hence, let us establish that y is binary at any vertex of Z_2 on the hyperplane (26b). That is, in addition to the active constraint (26b), let us explore three additional active constraints from the remaining inequalities that would yield a unique feasible solution.

Case (i): $x_1 = 0$ is active (the case of $x_2 = 0$ being active is symmetric).

Hence, from (26b) being assumed active, we have

$$x_2 = (\beta_2 - \alpha_2)y_1 + (\beta_2 - \alpha_1)y_2 + (\alpha_1 + \alpha_2 - \beta_2). \tag{27}$$

- If in addition, $x_2 = 0$ is active, then noting from (2) that the right-hand side in (27) must be positive, we have a contradiction.
- On the other hand, if $x_2 = \alpha_2 y_2$ is active, then we must have from (27) that

  $$(\beta_2 - \alpha_2)y_1 + (\alpha_1 + \alpha_2 - \beta_2)(1 - y_2) = 0. \tag{28}$$

  The additional linearly independent hyperplane must come from (26c), implying that y_1 or y_2 is binary, and the other y-variable is determined by (28). Noting from (2) that $(\beta_2 - \alpha_2) \geq 0$ and $(\alpha_1 + \alpha_2 - \beta_2) > 0$, if $y_2 = 0$ then (28) leads to a contradiction, and if $y_2 = 1$, then (28) implies that $y_1 = 0$. Likewise, if $y_1 = 0$, then (28) implies that $y_2 = 1$, and if $y_1 = 1$, then (28) yields

  $$y_2 = \alpha_1 / (\alpha_1 + \alpha_2 - \beta_2). \tag{29}$$

  However, note that $\beta_2 \geq \alpha_2$, whereby if $\beta_2 = \alpha_2$, then we have $y_2 = 1$, but if $\beta_2 > \alpha_2$, then $y_2 > 1$ (noting $\alpha_1 + \alpha_2 > \beta_2$), yielding infeasibility.
- Else, if neither $x_2 = 0$ nor $x_2 = \alpha_2 y_2$ is active, then x_2 is given by (27) while y is determined solely by (26c) and is therefore binary valued.

Case (ii): $x_1 = \alpha_1 y_1$ is active (the case of $x_2 = \alpha_2 y_2$ being active is symmetric).

Hence, from (26b) being assumed active, we have,

$$x_2 = (\alpha_1 + \alpha_2 - \beta_2)(1 - y_1) + (\beta_2 - \alpha_1)y_2. \tag{30}$$

- If either x_1 or x_2 is zero, then the proof follows from Case (i).
- If $x_2 = \alpha_2 y_2$ is also active, then (30) yields (noting $\alpha_1 + \alpha_2 > \beta_2$ by (2)) that $y_1 + y_2 = 1$, and then in concert with active constraints from (26c), we get binary values of y.

- Finally, if no other constraint from (26a) is active, then $x_1 = \alpha_1 y_1$, x_2 is given by (30), and y is determined solely from (26c), and is therefore binary valued. This completes the proof. □

The question that arises is whether for any $n \geq 3$ as well, if we were to incorporate the class of inequalities (10) for each $k = 2, \ldots, n$ within X_n, we would derive conv(X_n). The answer is negative, even for $n = 3$ as the following example illustrates.

Example 2

Consider X_3 as given by (19) in Example 1, and suppose that we construct Z_3 by incorporating the inequalities (10) for $k = 2$ and $k = 3$ as given respectively by (20) and (21):

$$Z_3 = \{(x, y):\ 0 \leq x_1 \leq 6y_1,\ 0 \leq x_2 \leq 7y_2,\ 0 \leq x_3 \leq 8y_3,\ x_1 + x_2 + x_3 \leq 11,\ x_1 + x_2 \leq 3y_1 + 4y_2 + 3,$$
$$x_1 + x_2 + x_3 \leq -4y_1 - 3y_2 + y_3 + 17, \text{ and } 0 \leq y_j \leq 1,\ \forall\, j = 1, 2, 3\}. \tag{31}$$

Note that while (19d) is implied by (20) and $y_j \leq 1$, $\forall\, j$, (19e) is not necessarily implied and is explicitly incorporated within (31). Now, consider the vertex of (31) formed by the intersection of the following six linearly independent hyperplanes (note that $Z_3 \subseteq R^6$):

$$y_1 = 0,\ x_1 = 0,\ y_2 = 1,\ x_2 = 7y_2,\ x_3 = 8y_3, \text{ and } x_1 + x_2 + x_3 = 11. \tag{32}$$

The system (32) yields the unique solution

$$x_1 = 0,\ x_2 = 7,\ x_3 = 4,\ y_1 = 0,\ y_2 = 1,\ y_3 = ½, \tag{33}$$

which is feasible to the remaining constraints in Z_3, and is hence a (fractional) vertex of Z_3. Therefore, $Z_3 \neq$ conv(X_3). In fact, the following valid inequality for X_3 deletes this fractional vertex:

$$x_1 + x_2 + x_3 \leq 10 + y_3. \tag{34}$$

Note that when $y_3 = 1$, this is precisely (19e), while when $y_3 = 0$, (19c) implies that we must have $x_3 = 0$, whence (34) asserts that $x_1 + x_2 \leq 10$, which is valid by (19d). Moreover, (34) deletes the solution (33) and dominates (21) because $(10 + y_3) \leq -4y_1 - 3y_2 + y_3 + 17$, i.e., $4y_1 + 3y_2 \leq 7$. Indeed, incorporating (34) within Z_3 (and deleting the constraint $x_1 + x_2 + x_3 \leq 11$, which is now implied), we obtain a set Z_3', say, where we can demonstrate that $Z_3' = \text{conv}(X_3)$. But more importantly, this example has revealed another class of valid inequalities that we expose in the following section.

3. Other Classes of Valid Inequalities

The following result presents a class of valid inequalities that is prompted by Example 2. This particular class of inequalities is equivalent to the special case of the (ℓ, S) inequality from Barany et al. (1984a, b) where $S = \{1, \ldots, \ell - 1\}$. We provide a simple independent proof for this result, and then discuss several other such classes of valid inequalities that can be derived following this same philosophy.

Proposition 4. The following are valid inequalities for X_n:

$$\sum_{j=1}^{k} x_j \leq (\beta_k - \beta_{k-1}) y_k + \beta_{k-1}, \quad \forall\, k = 3, \ldots, n. \tag{35}$$

Proof. Consider any $k \in \{3, \ldots, n\}$. Note that if $y_k = 0$, then $x_k = 0$ by (1a), whence (35) reduces to (1b) for the case of $k - 1$. On the other hand, if $y_k = 1$, then (35) is precisely (1b) for the case of k. This completes the proof. □

The inequality (35) can be conceived as a "*depth-one*" cut that examines a right-hand-side value predicated on the case of y_k being zero or one for the case of k. In a similar vein, we can derive a variety of cuts by designing a right-hand-side of (35) based on multiple binary variables. For example, the following result derives a "*depth-two*" cut for $k \geq 4$ based on exploring binary values of y_k and y_{k-1}. This cut is a special case of the bottleneck cover inequality of Atamturk and Munoz (2003) and of the submodular inequality of Wolsey (1989), and bears some relationship to other classes of capacitated inequalities of Pochet (1988) and the dynamic knapsack induced inequalities for capacitated lot-sizing by Marchand (1998).

Proposition 5. The following are valid inequalities for X_n:

$$\sum_{j=1}^{k} x_j \le (\beta_{k-1} + \beta'_k - \beta_k) + (\beta_k - \beta'_k) y_{k-1} + (\beta_k - \beta_{k-1}) y_k, \text{ for } k = 4, \ldots, n, \tag{36}$$

where,

$$\beta'_k = \min\{\beta_k,\ \beta_{k-2} + \alpha_k\}. \tag{37}$$

Moreover, (36) uniformly dominates (35) for $k \ge 4$.

Proof. Consider the following inequality, where β'_k is given by (37):

$$\sum_{j=1}^{k} x_j \le [\beta_k\, y_{k-1}\, y_k + \beta_{k-1}\, y_{k-1}(1 - y_k) + \beta'_k\, y_k(1 - y_{k-1}) + \beta_{k-2}(1 - y_{k-1})(1 - y_k)]_L. \tag{38}$$

Observe that for binary values of (y_{k-1}, y_k), exactly one binary product term on the right-hand-side of (38) is one, with the corresponding coefficient yielding a valid bound on $\sum_{j=1}^{k} x_j$. By (1a, b), this bound is clearly given by β_k when $(y_{k-1}, y_k) = (1, 1)$, by β_{k-1} when $(y_{k-1}, y_k) = (1, 0)$, and by β_{k-2} when $(y_{k-1}, y_k) = (0, 0)$. Finally, when $(y_{k-1}, y_k) = (0, 1)$, we have $x_{k-1} = 0$ by (1a), and then, $\sum_{j=1}^{k-2} x_j + x_k \le \min\{\beta_k,\ \beta_{k-2} + \alpha_k\} = \beta'_k$, as defined in (37), by virtue of (1a, b). This establishes the validity of (38).

Now, (38) is of the form

$$\sum_{j=1}^{k} x_j \le \beta_{k-2} + (\beta_{k-1} - \beta_{k-2}) y_{k-1} + (\beta'_k - \beta_{k-2}) y_k - y_{k-1,k}(\beta_{k-1} + \beta'_k - \beta_{k-2} - \beta_k). \tag{39}$$

Note that $(\beta_{k-1}+\beta'_k-\beta_{k-2}-\beta_k)=(\beta_{k-1}-\beta_{k-2})\geq 0$ when $\beta'_k=\beta_k$, and also, when $\beta'_k=\beta_{k-2}+\alpha_k$, we get $(\beta_{k-1}+\beta'_k-\beta_{k-2}-\beta_k)=(\beta_{k-1}+\alpha_k-\beta_k)>0$ by (2). Hence, using $-y_{k-1,k}\leq -y_{k-1}-y_k+1$ in (39), as given by (9), we get the valid inequality (36).

Moreover, observe that when $\beta'_k=\beta_k$, then (36) is precisely of the form (35). Otherwise, if $\beta'_k<\beta_k$, then (36) implies (35), because then, the right-hand-side of (35) minus that of (36) is given by $(\beta_k-\beta'_k)(1-y_{k-1})\geq 0$. This completes the proof. □

Likewise, for $k \geq 5$, we can derive depth-three cuts, and so on. Actually, as discussed in the next section, we can use an inductive process to generate entire convex hull representations for X_n, $n \geq 2$, in a higher-dimensional space.

4. Inductive Process for Generating the Convex Hull Representation for X_n

As a preliminary, consider the following general result that lays the groundwork for inductively constructing conv(X_n) for $n \geq 2$ in a higher dimensional space.

Proposition 6. Consider a mixed-integer set X defined in variables $(x, y) \in R^n \times B^m$ (i.e., n continuous variables x and m binary variables y), and suppose that for some suitably defined set $S \subseteq R^n \times B^m$ and for its complement $\overline{S}$ with respect to $R^n \times B^m$, we have that

$$Z_0 = \text{conv}(X \cap S) = \{(x, y): \; Ax + Dy \leq b\} \tag{40a}$$

and

$$Z_1 = \text{conv}(X \cap \overline{S}) = \{(x, y): \; Gx + Hy \leq g\} \tag{40b}$$

where (40a) and (40b) define bounded sets. Then,

conv(X) = $Z \equiv \{(x, y)$: for some $w \in R^n$, $v \in R^m$, and $0 \leq Y \leq 1$, we have

$$A(x-w)+D(y-v)\leq b(1-Y)$$

$$Gw + Hv \le gY\}. \tag{41}$$

Proof. First, let us establish that

$$\text{conv}(X) = \text{conv}(Z_0 \cup Z_1). \tag{42}$$

This follows readily by noting that $X \subseteq Z_0 \cup Z_1$, and so, $\text{conv}(X) \subseteq \text{conv}(Z_0 \cup Z_1)$. Conversely, since $X \cap S \subseteq X$, we have $Z_0 = \text{conv}(X \cap S) \subseteq \text{conv}(X)$, and similarly, $Z_1 \subseteq \text{conv}(X)$, and so, $Z_0 \cup Z_1 \subseteq \text{conv}(X)$, i.e., $\text{conv}(Z_0 \cup Z_1) \subseteq \text{conv}(X)$. Hence, (42) holds true.

By the disjunctive convex hull generation process of Balas (1998), (see also Balas (1979) and Sherali and Shetty (1980)), or the RLT process of Sherali and Adams (1990, 1994), we can construct conv(X) via (42) by multiplying (40a) by $(1 - Y)$ and (40b) by Y, where $0 \le Y \le 1$, and then using the substitutions $w = [x\,Y]_L$, $v = [y\,Y]_L$. This yields (41), and the proof is complete. □

An important specialization of Proposition 6 is given by the following result.

Corollary 1. In Proposition 6, suppose that

$$S = \{(x, y) \in R^n \times B^m\text{: at least one } y_i = 0 \text{ for } i = 1,\ldots,m\}\text{, and} \tag{43a}$$

$$\overline{S} = \{(x, y) \in R^n \times B^m\text{: } y_i = 1,\ \forall\, i = 1,\ldots,m\}. \tag{43b}$$

Accordingly, let Z_0 and Z_1 defined in (40a, b) be given by

$$Z_0 = \{(x, y)\text{: } Ax + Dy \le b\}\text{, and } Z_1 = \{(x, y)\text{: } Gx \le g,\ y_i = 1,\ \forall\, i = 1,\ldots,m\}, \tag{44}$$

where each of these sets is bounded. Then,

$$\text{conv}(X) = \{(x, y)\text{: for some } w \in R^n,\ 0 \le Y \le 1\text{, we have}$$

$$A(x - w) + D(y - eY) \le b(1 - Y)$$

$$Gw \le gY\}, \tag{45}$$

where $e = (1,\ldots,1)^T \in R^m$.

Proof. Adopting (42), and multiplying the constraints defining Z_0 and Z_1 in (44) by $(1 - Y)$ and Y respectively, we get upon substituting $w = [x\ Y]_L$ and $v = [y\ Y]_L$ that

$$\text{conv}(X) = \{(x, y):\ A(x - w) + D(y - v) \le b(1 - Y)$$

$$Gw \le gY,\ v_i = Y,\ \ \forall\ i = 1,\ldots,m\}. \tag{46}$$

Eliminating v from (46) by substituting $v = (e)Y$, we get (45). This completes the proof. □

Remark 3. Notice in (45) of Corollary 1 that when $Y = 1$, by the boundedness assumption of Z_0 (that would preclude recession directions, i.e., nonzero solutions to the corresponding homogeneous system), we have, $x = w$ and $y = (e)Y$, and so, $(x, y) \in Z_1$. Likewise, when $Y = 0$, we get by the boundedness of Z_1 that $w = 0$, and $(x, y) \in Z_0$. As such, the variable Y is playing the role of $\left[\prod_{i=1}^{n} y_i\right]_L$. □

To illustrate the application of Proposition 6 and Corollary 1, let us first consider the case $n = 2$, and then inductively demonstrate how one could handle the case of $n = 3$. Further generalizations or extensions would then be evident.

For the case of $n = 2$, applying the special case of Corollary 1 with S and $\overline{S}$ given by (43), we get from (40) and (44) that

$$Z_0 = \{(x, y):\ 0 \le x_1 \le \alpha_1 y_1,\ 0 \le x_2 \le \alpha_2 y_2,\ y_1 + y_2 \le 1,\ y \ge 0\} \tag{47a}$$

and

$$Z_1 = \{(x, y):\ 0 \le x_1 \le \alpha_1,\ 0 \le x_2 \le \alpha_2,\ x_1 + x_2 \le \beta_2,\ y_1 = y_2 = 1\}. \tag{47b}$$

Observe that $Z_0 = \text{conv}(X_2 \cap S)$ since $x_1 + x_2 \le \beta_2$ is redundant under the condition $\{y_1 = 0$ or $y_2 = 0\}$, because $\beta_2 \ge \max\ \{\alpha_1, \alpha_2\}$ by (2), and moreover, y is readily verified to be binary valued at all vertices of Z_0. Hence, noting that $Y \equiv y_{12}$ as in Remark 3, we can write the system (45) as follows:

$$\text{conv}(X_2) = Z \equiv \{(x, y)\text{: for some } w_1, w_2, \text{ and } 0 \le y_{12} \le 1, \text{ we have,}$$

$$0 \le (x_j - w_j) \le \alpha_j(y_j - y_{12}) \text{ for } j = 1, 2 \tag{48a}$$

$$y_{12} \le y_j \text{ for } j = 1, 2, \text{ and } y_{12} \ge y_1 + y_2 - 1 \tag{48b}$$

$$0 \le w_j \le \alpha_j y_{12} \text{ for } j = 1, 2 \tag{48c}$$

$$w_1 + w_2 \le \beta_2 y_{12}\}. \tag{48d}$$

Moreover, as shown below, the set Z, which is the projection of the higher dimensional set (48) onto the original (x, y) variable space, indeed yields the set Z_2 given by (26), thereby verifying Proposition 3.

Proposition 7. $Z = Z_2$, where Z and Z_2 are given by (48) and (26), respectively.

Proof. First, let us verify that $Z \subseteq Z_2$, by demonstrating that the constraints of Z_2 are implied by Z. Observe that (48a) and (48c) yield (26a). Also, (48b) along with $0 \le y_{12} \le 1$ yield (26c). Finally, the constraint (26b) results from (48) by surrogating (48a) for $j = 1, 2$, and using (48d) to get

$$(x_1 + x_2) \le (w_1 + w_2) + \alpha_1(y_1 - y_{12}) + \alpha_2(y_2 - y_{12}) \le \alpha_1 y_1 + \alpha_2 y_2 - y_{12}(\alpha_1 + \alpha_2 - \beta_2).$$

Now, using $-y_{12} \le -y_1 - y_2 + 1$ from (48b), and that $\alpha_1 + \alpha_2 > \beta_2$ by (2), we get (26b).

Conversely, to verify that $Z_2 \subseteq Z$, it is sufficient to show that every vertex of Z_2 has a completion w_1, w_2, and y_{12} that is feasible to (48). But by Proposition 3, we know that the vertices of Z_2 have binary values of y. Hence, given $(x, y) \in \text{vert}(Z_2)$, by taking $y_{12} \equiv y_1 y_2$, $w_1 \equiv x_1 y_1$, and $w_2 \equiv x_2 y_2$, we readily verify that this yields a feasible solution to Z. This completes the proof. □

To apply the tool of Proposition 6 inductively, consider X_3. We can write

$$\mathrm{conv}(X_3) = \mathrm{conv}(Z_0 \cup Z_1) \tag{49}$$

where,

$$Z_0 = \mathrm{conv}[X_3 \cap \{(x, y): \text{ at least one } y_i = 0 \text{ for } i = 1, 2, 3\}] \tag{50a}$$

and

$$Z_1 = \mathrm{conv}[X_3 \cap \{(x, y): \ y_i = 1, \ \forall\ i = 1, 2, 3\}]$$

$$\equiv \left\{(x, y): \ 0 \le x_j \le \alpha_j \text{ for } j = 1,2,3, \sum_{j=1}^{k} x_j \le \beta_k \text{ for } k = 2,3, \ y_i = 1 \text{ for } i = 1,2,3\right\}. \tag{50b}$$

For describing Z_0, so that we could then apply Proposition 6, we use this Proposition 6 in a nested form itself by writing

$$Z_0 = \mathrm{conv}(Z_{00} \cup Z_{01}) \tag{51}$$

where,

$$Z_{00} = \mathrm{conv}[X_3 \cap \{(x, y): \ y_3 = 0\}] \tag{52a}$$

and

$$Z_{01} = \mathrm{conv}[X_3 \cap \{(x, y): \ y_3 = 1 \textit{ and } \text{at least one of } y_1 \text{ and } y_2 \text{ is zero}\}]. \tag{52b}$$

Observe that Z_{00} is given by Z_2 of (26) for the case of $n = 2$, while

$$Z_{01} = \mathrm{conv}[(x, y): \ 0 \le x_j \le \alpha_j y_j \text{ for } j = 1, 2, 0 \le x_3 \le \alpha_3, x_1 + x_2 + x_3 \le \beta_3,$$
$$y_1 + y_2 \le 1, y_3 = 1, y \text{ binary}]. \tag{53}$$

This set Z_{01} can now be constructed by applying the special GUB structured RLT process described in Sherali et al. (1998), and then working backwards, we can derive $\mathrm{conv}(X_3)$ by this process.

While this mechanism is generalizable for any n in theory, in practice, it can be applied to relaxations of the type X_2 and X_3, say, in order to generate tighter higher dimensional reformulations whose projections could potentially capture several classes of valid inequalities. In addition, such constructs can be augmented by valid inequalities as prescribed by Propositions 1, 2, 4, and 5, as well as others that are described in the literature for the single item capacitated lot-sizing problem as in Pochet (1988), Marchand (1998), Wolsey (1989), Loparic et al. (2003), and Atamturk and Munoz (2003). In particular, while Propositions 1, 4, and 5 recover certain special cases of flow cover, (ℓ, S), submodular, and bottleneck cover inequalities using RLT-based lifting arguments, it is of interest to explore if this viewpoint might offer a unifying framework for generating the aforementioned classes of inequalities in general. As another topic of future research, it is worthwhile to study if the higher dimensional convex hull representations afforded by RLT might reveal new classes of valid inequalities in the original variable space based on characterizing specific extreme directions of the dual projection cone (see Sherali et al. (1995) for an illustration of this approach in the context of the Boolean quadric polytope). Finally, we propose for future research to conduct a computational study of applying such cuts to practical problems that have an embedded nested fixed-charge structure as described by X_n in (1).

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