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Parametric Branch and Bound

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The paper describes a procedure for mixed integer programming that allows branches to be imposed 'by degrees', which can subsequently be revised or weeded out according to their relative influence. It is an adaptive approach in which the branch and bound tree can be manipulated and restructured. The approach also yields measures of the costs of imposing the branches that lead to integer solutions, thus providing a built-in form of sensitivity analysis for evaluating the effect of integer restrictions.

1. INTRODUCTION

THIS paper introduces a parametric branch and bound (B&B) procedure that has greater flexibility than ordinary B&B. Once a branch is taken in ordinary B&B, it is largely irrevocable—i.e. all descendant branches must inherit the limitations imposed by their predecessors. In parametric B&B, a descendant branch may partly or wholly undo an antecedent branch. Moreover, once a feasible mixed integer programming (MIP) solution is obtained, then the 'actual' branches that achieved this solution can be identified by weeding out uninfluential branches created during the solution process.

The strategy underlying parametric B&B is to incorporate variables and constraints into the objective function in a manner resembling 'multi-objective' or 'goal programming' approaches [1, 2, 9]. The process is especially direct for 0–1 MIP problems, which do not require the creation of additional variables. However, even in the general case, all calculations can be carried out relative to a compact basis that is the same size as for the original problem.

The use of weighted variables and constraints in the objective function also bears a connection to 'Lagrangean' approaches [3, 6, 10]. However, in contrast with Lagrangean techniques, the weights are not designed to solve a dual problem, but rather entirely overshoot dual feasibility. The process may be viewed as that of constructing tentative duals (at least implicitly), guided by considerations relevant to the B&B setting. Also, in contrast to standard B&B and its exploitation of duality, parametric branches are conveniently handled by postoptimizing with the primal simplex method, whereas ordinary branches are often preferably handled by postoptimizing with the dual simplex method. Instead of imposing a branch either fully or not at all, parametric B&B allows one to impose branches 'by degrees'. This is extremely important for enabling branching alternatives to be carefully analyzed—and revised, if desirable—at later stages of the tree.

By its nature, parametric B&B yields information about the cost of imposing branches that lead to integer solutions. This information can then be used in a sensitivity analysis for evaluating the significance of integer restrictions. This type of analysis can also be coupled with an approach that attaches penalties to deviations from constraints. Such an approach yields a combined sensitivity analysis characterizing the influence of integer assignments on problem constraints.

2. PARAMETRIC B&B FOR 0–1 PROBLEMS

The basic ideas of parametric B&B are relatively straightforward for 0–1 MIP problems, and we first examine them in this setting. For our purposes, the 0–1 problem will be written

$$\text{Minimize } x_0 = \sum_{j\epsilon N} c_j x_j \qquad (2\text{-}1)$$

$$\sum a_{ij} x_j \leq b_i \qquad i\epsilon M \quad (2\text{-}2)$$

$$x_j \geq 0 \qquad j\epsilon M \quad (2\text{-}3)$$

$$1 \geq x_j \geq 0 \qquad j\epsilon J \quad (2\text{-}4)$$

$$x_j \quad \text{integer} \qquad j\epsilon J \quad (2\text{-}5)$$

where J is the index set of the integer variables and is a subset of N. In our discussion of 0–1 problems, we will confine ourselves to elaborating the principal ideas of parametric B&B, together with illustrating the more rudimentary types of considerations. In fact, most of the implementation aspects of the 0–1 case require no commentary other than to indicate the relevant decision alternatives. Subsequently, more advanced aspects of implementation will be introduced for the general MIP case, and then linked to the 'sensitivity analysis' framework.

In the 0–1 setting, consider any two disjoint subsets J_0 and J_1 of J and the 'contrived' objective function

$$\text{Minimize } z_0 = d_0 + \sum_{j\epsilon J_0} d_j x_j - \sum_{j\epsilon J_1} d_j x_j \qquad (2\text{-}1')$$

where all of the d_j are positive, and where

$$d_0 = \sum_{j\epsilon J_1} d_j.$$

By the form of z_0, if there exists a feasible solution to the LP problem (2-1)–(2-4) that satisfies $x_j = 0$ for $j\epsilon J_0$ and $x_j = 1$ for $j\epsilon J_1$ then this will be an optimal solution to the LP problem in which (2-1′) replaces (2-1). (In fact, this assignment of values to the x_j for $j\epsilon J_0 U J_1$ is uniquely optimal when feasible, and hence must occur at an LP extreme point. Note that this gives a rather simple proof of the fact that feasible integer assignments are unique extreme points of the pure 0–1 problem and occur at one or more extreme points of the mixed 0–1 problem.)

In addition, by the definition of d_0, it follows that $z_0 \geq 0$ for all feasible LP solutions, and $z_0 = 0$ only for a solution that yields the assignment $x_j = 0$ for $j\epsilon J_0$ and $x_j = 1$ for $j\epsilon J_1$. Consequently, the creation of a composite objective function

$$\text{Minimize } u_0 = x_0 + z_0 \qquad (2\text{-}1'')$$

assures by the non-negativity of z_0 that

$$\text{Min } u_0 \geq \text{Min } x_0.$$

Drawing on the fact that the constraint $z_0 = 0$ is equivalent to the 'partial assignment' $x_j = 0$, $j\epsilon J_0$ and $x_j = 1$, $j\epsilon J_1$ we may use the objective (2-1″) as a device to compel this partial assignment to hold. Clearly, if the parameters d_j are selected large enough, then $z_0 = 0$ must result provided the corresponding partial assignment is feasible.

Handling the composite objective function in parametric B&B

The first stage of the parametric B&B approach for 0–1 problems utilizes these ideas to influence the creation of partial assignments through manipulation of the parameters d_j. However, instead of assigning these parameters preemptively large values, more moderate values are assigned and then monitored in order to determine interactive effects relevant to the B&B setting. By the familiar Lagrangean type of argument, we may observe

$$\text{Min } \{x_0 \text{ subject to } z_0 = 0\}$$
$$= \{\text{Min } u_0 \text{ subject to } z_0 = 0\} \geq \text{Min } u_0$$

and hence the composite objective function (2-1″) always yields a lower bound on the ordinary B&B objective (in which $z_0 = 0$ is explicitly imposed). This type of bound information can be used for fathoming in a manner resembling that of the ordinary B&B approach. We will discuss the way to accommodate fathomed alternatives in the parametric setting after introducing the notion of a 'parametric branch'.

0–1 *Parametric branching*

The branch step for the 0–1 parametric approach is a 'tentative' operation that either becomes consolidated or revised on the basis of information subsequently generated. Quite simply, if the branch corresponds to the assignment $x_r = 0$ then the current objective u_0 is updated to become $u_0 + d_r x_r$ and if the branch corresponds to the assignment $x_r = 1$ then u_0 is updated to become $u_0 + d_r(1 - x_r)$ or equivalently $u_0 + d_r x_r'$ where x_r' is the slack variable for the inequality $x_r \leq 1$. The weight d_r is selected so that the updated representation of u_0 is dual infeasible, and therefore, allows reoptimization with the primal simplex method.

To illustrate, suppose the current LP representations of u_0 and x_r are given by

$$u_0 = 12\tfrac{1}{4} + 5\tfrac{1}{2}\, x_3 + 1\tfrac{1}{4}\, x_4 + 4x_5,$$
$$x_r = \tfrac{1}{4} - 2x_3 + x_4 + \tfrac{3}{4}x_5.$$

Then for the branch $x_r = 0$ any value of d_r satisfying $d_r > 5\frac{1}{2}/2$ will cause the new objective $u_0 + d_r x_r$ to be dual infeasible. (The ratio $5\frac{1}{2}/2$ is exactly the pivot ratio for the dual simplex method that would be identified if one were to introduce the constraint $x_r \leq 0$.) Thus, for example, selecting $d_r = 3$, we obtain the new u_0 objective

$$u_0 = 13 - \tfrac{1}{2}\, x_3 + 4\tfrac{1}{4}\, x_4 + 6\tfrac{1}{4}\, x_5.$$

The negative coefficient for x_3 of course signals that re-optimization may now be undertaken with the primal simplex method.

Alternatively, for the branch $x_r = 1$ a value of d_r satisfying $d_r > 1\frac{1}{4}/1$ will succeed in establishing dual infeasibility for the objective $u_0 + d_r x'_r$ where $x'_r = 1 - x_r$. (Here $1\frac{1}{4}/1$ is the pivot ratio for the dual simplex method relative to the constraint $x_r \geq 1$.) Thus, for example, selecting $d_r = 2$ we obtain the new objective

$$u_0 = 13\tfrac{3}{4} + 9\tfrac{1}{2}\, x_3 - \tfrac{3}{4}\, x_4 + 2\tfrac{1}{2}\, x_5.$$

Values of d_r such as those selected in these examples may not be sufficiently large to insure that the branches for $x_r = 0$ and $x_r = 1$ will utlimately be enforced. However, as previously noted, the procedure does not seek preemptive values but rather seeks to analyze the consequences of more moderate values. (In this connection, it is easily established that if a dual pivot on the constraint $x_r \leq 0$ or $x_r \geq 1$ would achieve primal feasibility in one step, then any value of d_r that exceeds the dual pivot ratio will achieve exactly this same solution in one primal iteration.)

This extremely simple type of 'local' implementation step for 0–1 parametric B&B has a direct analog in a variety of standard 'parametric' approaches for ordinary linear programming. The crucial aspect in the B&B setting is the way in which this step is used—i.e. the manner in which the parametric branches are processed to yield information and branching alternatives not available to ordinary B&B. Thus, in particular, parametric branches of the form just illustrated are not regarded as ironclad impositions nor are they initially assigned a sequential ranking. Indeed, it may well occur at a subsequent step that an 'earlier' branch will be discovered to be superfluous in terms of other more influential branches. We discuss this aspect of the approach next.

Revised and augmented parametric branches

Branches that are currently uninfluential can easily be singled out by the parametric B&B procedure as follows. Suppose the updated LP representation of u_0 gives x_r or x'_r (depending on which of the two variables was previously assigned a weight) a coefficient that equals or exceeds d_r. (The implication, of course, is that x_r or x'_r is currently nonbasic.) Then, reducing this coefficient by d_r still leaves the objective dual feasible. This means that the parametric branch *can be eliminated without changing the current LP solution.*

The step of weeding out uninfluential branches (i.e. those subject to elimination) can be postponed until a feasible 0–1 solution is obtained. Then, all variables whose parameter values can be reduced to zero may be excluded from the branching category. (An exception occurs for branches that are antecedents of a compulsory branch, as described in the next section.) The remaining variables and parametric branches may further be ranked, for example, according to the magnitude of the d_r values that are required to make the current solution optimal. In this fashion, the 'actual' branches of the B&B process, and their sequence, are decided upon at a stage in which the true significance of these branches can more accurately be assessed.

Another means for evaluating the relative influence or significance of a branch occurs when a branching variable 'resists' its weight and becomes basic. The decision must then be whether to increase or decrease d_r. (The latter may be a compounded step that both decreases d_r to 0 for the current parametric branch and then assigns d_r a positive value for the alternative branch.) By means of such decisions, earlier parametric branches may either be revised or reinforced.

The technical aspects of implementing these ideas are relatively simple, and hence, we confine ourselves to the task of discussing only the more prominent strategic considerations, deferring the particulars of implementation to the general MIP case, where they are more critical.

Compulsory branches and sequences

The minimization of u_0 as noted earlier provides a lower bound on the optimum value of x_0 subject to the partial assignment associated with $z_0 = 0$. Whenever this bound equals or exceeds the value of x_0 for the best MIP solution currently known (which automatically occurs when the current LP solution in fact provides this best MIP solution), then all further continuations of the partial assignment are identified as unproductive in the usual B&B sense. This immediately provides the option of 'back-tracking'—i.e. of deciding which parametric branch underlying the current LP solution should be considered the 'last branch'. The observation that the determination of the last branch can be deferred to yield a more flexible variant of the LIFO procedure was first made by Tuan [11]. In the current approach, this determination can be based, for example, on the current d_j values (after weeding out uninfluential branches). Thus, in this fashion, a sequential ordering is gradually imposed on the branches.

Having identified a last branch, the alternative branch is now imposed as *compulsory*. This means that the method does not allow this branch to be reversed or discarded (following the usual backtracking rules), until the current partial assignment is fathomed. In particular, as soon as a compulsory branch is identified, all other parametric branches that are currently in force (i.e. all those for which d_j is currently positive) must be considered as prior to the compulsory branch. This does not impose any particular sequence on these prior branches (until backtracking again compels one to be identified as a 'last member'), but does impose *induced bounds* on the parameters d_j. That is, these positive d_j coefficients cannot be reduced below their current values as long as the compulsory branch remains in effect.

There are, however, three important exceptions to the strict maintenance of induced bounds. The first and obvious exception occurs by deciding upon some partial sequence for the parametric branches and employing a tree search rule that jumps back over some of the 'later' branches of this sequence. Then, the induced bounds can be disregarded for the branches thus bypassed. (The same remark holds for branches that are released or reversed as a result of carrying out a backtracking step with the LIFO rule.) The second exception occurs by redeciding the status of the branch that has been designated compulsory. That is, a compulsory branch may be revised, if the reconfigured objective u_0 that results after a series of additional iterations makes this appear eminently desirable. (Such a process must, of course, be sufficiently systematized to guard against circularity.) Finally, just as compulsory branches may be imposed as standard branches rather than parametric branches (as by the use of pre-emptive weights), so may the identified antecedents of the compulsory branches be imposed in the standard fashion. This imposition, however, does not carry with it an implied sequence for the antecedents themselves. Further, greater flexibility is achieved by monitoring the values of parameters that would suffice to maintain the imposed branches, thereby retaining the option of amending the status of these branches at subsequent stages.

3. PARAMETRIC B&B FOR GENERAL MIP PROBLEMS

Many of the same strategic notions discussed in the preceding section carry over to the general MIP problem. However, there is a very significant difference between the general problem and the 0–1 problem. Specifically, in the general case, there may not exist weights that will cause all of the branching inequalities to be satisfied. Moreover, if such weights exist, they may be very hard to find, or cause some inequalities to be unnecessarily over-satisfied, thus failing to identify integer solutions. We now examine the special techniques that permit these difficulties to be overcome and allow the general approach to be implemented successfully.

Branching schemes for the general MIP problem may be viewed as the successive imposition of upper and lower bounds on the problem variables. Thus, at any particular stage, the integer variables are governed by restrictions of the form

$$U_j \geq x_j \geq L_j \qquad j \epsilon J \tag{3-1}$$

where U_j and L_j may represent original bounds on x_j or those currently inherited by branching. The general parametric approach seeks to incorporate such bound restrictions into the objective function in a manner that will enable the same types of evaluations and manipulations that are possible in the 0–1 setting.

This is accomplished by introducing non-negative variables z_j and the inequalities

$$U_j + z_j \geq x_j \geq L_j - z_j \qquad j\epsilon J. \tag{3-2}$$

Thereupon, the z_j variables are incorporated into the minimization of x_0 to create the composite objective function

$$\text{Minimize } u_0 = \sum_{j\epsilon N} c_j x_j + \sum_{j\epsilon J} d_j z_j. \tag{3-3}$$

It is clear that for the weights d_j sufficiently large all of the z_j will be 0 and (3-2) will reduce to (3-1)—provided, of course, that (3-1) is consistent with the original problem constraints. Also, for $d_j > 0$ we have

$$\text{Min } u_0 \geq \text{Min } x_0$$

and

$$\text{Min } \{x_0 \text{ subject to (3-1)}\} = \text{Min } \{u_0 \text{ subject to (3-1)}\} \geq \text{Min } u_0.$$

Thus, the same bound relationships hold in the general MIP settings as in the 0–1 context upon introducing (3-2) and (3-3).

The problem at hand is how to manage the system (3-2) and (3-3) effectively. This is extremely important because otherwise, the introduction of (3-2) and (3-3) doubles the number of integer variables and adds a corresponding number of new constraints. The following approach for dealing with this problem constitutes an adaptation of procedures for 'weighted deviation' problems introduced in [7].

Let s'_j and s''_j denote slack variables for the inequalities of (3-2), yielding the equations

$$x_j - z_j + s'_j = U_j, \tag{3-4}$$
$$-x_j - z_j + s''_j = -L_j.$$

Adding these two equations, we obtain

$$-2z_j + s'_j + s''_j = U_j - L_j. \tag{3-5}$$

Equation (3-5) identifies a 'primal relationship' between the variables z_j, s'_j and s''_j. On the 'dual' side, let u_j, v'_j and v''_j represent dual variables associated with the non-negativity restrictions on z_j, s'_j and s''_j respectively. (That is, the defining equations for u_j, v'_j and v''_j are respectively given by the z_j, s'_j and s''_j columns of the primal.) Then, since the coefficients of z_j, s'_j and s''_j are d_j, 0 and 0 in (3-3), it follows from (3-4) that the relationship between the columns for z_j, s'_j and s''_j can be summarized by

$$u_j + v'_j + v''_j = d_j. \tag{3-6}$$

This relationship, in conjunction with (3-5), provides the key to implementing the general paramatric B&B approach without modifying the size of the LP tableau. In particular, it is readily established that *at least one of the variables z_j, s'_j and s''_j must be basic and at least one must be nonbasic.*

Consequently, only one of the three primal variables need be explicitly included in the tableau at any given time, whereupon the form of the other variables is always known from (3-5) and (3-6). Similarly, x_j need not be included in the tableau since it is always capable of being recovered from an equation of (3-4). The fundamental relationships of the approach as they apply in the present setting can be summarized as follows. (Formal proofs of these relationships are omitted, since their derivation is a direct consequence of (3-5) and (3-6).)

(R.0)—Of the two variables (from the group z_j, s'_j and s''_j) that are not explicitly included in the LP tableau, one is currently basic and one is currently nonbasic.

(R.1)—*If the explicit variable is nonbasic*:

(a) The tableau row for the implicit basic variable is precisely the primal equation (3-5).

(b) The tableau column for the implicit nonbasic variable is the negative of the column for the explicit variable, except that the objective function coefficient for the implicit variable is d_j minus the objective function coefficient of the explicit variable, and the column coefficient that corresponds to the implicit basic variable is as given in (a).

(R.2)—*If the explicit variable is basic*:

(a) The tableau column for the implicit nonbasic variable is precisely the dual equation (3-6). Thus, the objective function coefficient is d_j and the column has unit coefficients in the positions corresponding to its two companion basic variables, with 0's elsewhere.

(b) The tableau row for the implicit basic variable is obtained by substituting the current expression for the explicit basic variable in (3-5) and giving the implicit nonbasic variable a unit coefficient [as in (a)].

We can now give the rules for implementing the parametric B&B approach that result from these relationships. After stating these rules, we will illustrate their use by numerical example.

1. The initial step solves the LP problem without the additional variables and constraints of (3-4).

2. The parametric branching step introduces the restrictions (3-4) in a tableau for which x_j is basic and does not satisfy $U_j \geq x_j \geq L_j$.

   (a) Identify the equation of (3-4) which implies $z_j > 0$ for s'_j and s''_j nonnegative (i.e. the equation corresponding to the 'violated bound');
   (b) Omit the slack variable s'_j or s''_j from the equation identified in (a), and use this reduced equation to replace x_j with z_j by direct substitution. The omitted slack variable is the implicit nonbasic variable, and slack variable of the other equation of (3-4) is the implicit basic variable. (These implicit variables are irrelevant at this point). The variable z_j is now basic.
3. The next component step of parametric branching is to create (or increase) the weight d_j for a selected variable z_j that is currently basic. To do this, add d'_j times the current z_j equation to the objective function equation, where d'_j denotes the increment in d_j, selected large enough to require re-optimization with the primal method. (The effect of this step on any currently implicit nonbasic variables is immediately given in relation (R.1)(b), since none of the implicit nonbasic variables governed by (R.2)(a) are affected except for the companion of z_j whose coefficient is always d_j.)
4. An arbitrary iteration of the primal method can be implemented by the following observations:

   (a) An implicit nonbasic variable provides an eligible pivot column if and only if its explicit companion is nonbasic and has a coefficient exceeding d_j (as a result of relations (R.2)(a) and (R.1)(b)).
   (b) If an implicit nonbasic variable is selected for the pivot column, then it can replace the previous explicit variable, which now becomes implicit.
   (c) No implicit basic variable need be considered in determining the pivot row unless both slack variables s'_j and s''_j are basic. If an implicit basic variable provides the pivot row, it replaces the currently explicit basic variable as the new explicit variable.

Condition 4(c) is a direct consequence of (3-5) and implies that once a slack variable of (3-4) first becomes implicitly basic, it remains irrelevant unless the other slack becomes basic—or until a new branch is introduced by tightening one of the U_j or L_j values. Branching can occur only if both of the slacks s'_j and s''_j are basic and positive (since otherwise x_j would be assigned an integer value or lie outside one of its provisional current bounds) and thus, in this case, one can simply jettison all three of the s'_j, s''_j and z_j variables [after recovering x_j from (3-4)] and start again with Step 2.

In accordance with these observations, if either U_j or L_j is a 'true' bound for x_j, then this bound can be rigidly enforced simply by rendering z_j an ineligible pivot column (e.g. by increasing d_j) whenever the slack variable associated with the true bound is nonbasic. (If z_j becomes basic when the alternative slack is nonbasic, (3-5) implies that x_j must satisfy its appropriate bound, and condition 4(c) assures that the critical slack will not become nonbasic unless z_j is nonbasic.)

We now illustrate these considerations with the following numerical example.

Example

Consider the MIP problem represented by the following 'condensed' tableau format, where x_1 and x_2 are the integer variables.

			x_2	
$x_0 =$		-9	-4	7
$x_4 =$		-5	-2	3
$x_5 =$		1	-3	-3

The problem constraint equations are read directly from the tableau (e.g. $x_4 = 7 - 5x_1 - 2x_2 + 3x_3$). Pivoting on the circled element with the primal simplex method yields the optimal LP tableau

		x_1	x_4	x_3
$x_0 =$			2	
$x_2 =$	$3\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
$x_5 =$	$1\frac{1}{2}$	$\frac{17}{2}$	$\frac{3}{2}$	$-\frac{15}{2}$

Suppose now that we wish to branch parametrically on the inequality $x_2 \leq 3$ by Step 2 of the preceding rules, thus obtaining $U_2 = 3$ and $L_2 = 0$ (where $L_2 = 0$ represents a true lower bound). The first equation of (3-4) implies $z_2 > 0$ and yields $z_2 = x_x - U_2$ upon omitting the slack variable s'_2. Thus, the tableau equation for z_2 is given by

		x_1	x_4	x_3
$z_2 = x_2 - 3 =$	$\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

Next, by Step 3, we add a sufficient multiple of this equation to the objective equation to create dual infeasibility. Any value exceeding the 'dual pivot ratio' 1/(5/2) will do, and we select a multiple of 1 to give $u_0 = x_0 + 1z_2$ or

		x_1	x_4	x_3
$u_0 =$	$-13\frac{1}{2}$	$-\frac{3}{2}$	[illegible]	$\frac{5}{2}$

Thus, putting Steps 2 and 3 together (replacing the x_2 equation by the z_2 equation and replacing x_0 by u_0) we obtain the tableau

		x_1	x_4	x_3
$u_0 =$	$-13\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$z_2 =$	$\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
$x_5 =$	$1\frac{1}{2}$	$\frac{17}{2}$	$\frac{3}{2}$	$-\frac{15}{2}$

The primal simplex method now pivots on the circled element to obtain

		z_2	x_4	x_3
$u_0 =$		$\frac{3}{5}$	$\frac{9}{5}$	$\frac{8}{5}$
$x_1 =$	$\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$
$x_5 =$	$\frac{16}{5}$	$-\frac{17}{5}$	$-\frac{1}{5}$	$-\frac{12}{5}$

Since optimality is re-established in a single pivot, this is the same tableau that would have been obtained by a 'dual' pivot on the constraint $x_2 \leq 3$ (except for the z_2 column). With z_2 and s'_2 both nonbasic (the first explicitly, the second implicitly), we may infer that the current value of x_2 is 3 (as seen by (3-4) with $U_2 = 3$). Consequently, at the present stage the parametric branch has achieved the same effect as an ordinary branch, although we do not assign the branch a sequential 'slot' and leave it open to subsequent revision.

Suppose that $x_1 \geq 1$ is selected to provide the next parametric branch. By Step 2, the relevant expression for z_1 is $z_1 = L_1 - x_1 = 1 - x_1$ giving the tableau representation

		z_2	x_4	x_3
$z_1 =$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$-\frac{3}{5}$

By Step 3, we again seek to add a sufficient multiple of z_1 to the current objective function to produce a dual infeasibility. Since the implicit nonbasic variable s'_2 has a coefficient which is the negative of the explicit z_2 coefficient (i.e. $-\frac{2}{5}$), we must take the effect of s'_2 into account in the manner noted parenthetically in Step 3. By relation (R.1)(b), the current objective function coefficient of z_2 is $1 - \frac{3}{5} = \frac{2}{5}$. This yields the 'expanded' current u_0 and z_1 rows as follows

		z_2	x_4	x_3	s'_2
$u_0 =$	$-13\frac{4}{5}$	$\frac{3}{5}$	$\frac{9}{5}$	$\frac{8}{5}$	[illegible]
$z_1 =$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$-\frac{3}{5}$	[illegible]

From this, we see that a multiple of 2 suffices to produce dual infeasibility thereby giving the new objective function representation

		z_2	x_4	x_3	s'_2
$u_0 =$	$-12\frac{1}{5}$	$\frac{7}{5}$	$\frac{11}{5}$	$\frac{2}{5}$	$-\frac{2}{5}$

Note that the 'expansion' used in this example was not really necessary since the fact that z_2 has a positive coefficient immediately implies that we consider the implicit variable instead of—rather than in addition to—the explicit variable.

Because s'_2 gives the eligible pivot column, we replace z_2 by s'_2 (so that z_2 now becomes implicit) to yield the tableau

		s'_2	x_4	x_3
$u_0 =$	$-12\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$	$\frac{2}{5}$
$z_1 =$	$\frac{4}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	$-\frac{3}{5}$
$x_5 =$	$\frac{16}{5}$	$\frac{17}{5}$	$-\frac{1}{5}$	$-\frac{12}{5}$

Re-optimization with the primal method gives

		z_1	x_4	x_3
$u_0 =$	-13	1	2	1
$s'_2 =$	2	$-\frac{5}{2}$		[illegible]
$x_5 =$	10	$-\frac{17}{2}$		[illegible]

The variables of (3-4) associated with x_1 and x_2 have integer values in this tableau, and hence so do x_1 and x_2. Specifically, direct application of (3-4) yields $x_1 = 1$ and $x_2 = 1$ (noting that s''_1 and z_2 are implicitly nonbasic). Also, since z_1 and z_2 are both in the current basic solution, it follows that u_0 and x_0 have the same value.

We now analyze the parametric branches that provided this current integer solution. By relation (R.2)(a), the objective function coefficient of z_2 is d_2. Thus, reducing the coefficient of z_2 by d_2 (which effectively removes z_2 as a weighted variable in the objective function) still yields the same LP optimum. By our earlier observations for the 0–1 case, this means that the branching restriction that gave rise to z_2 is currently 'uninfluential', and thus we can shrink the B&B tree to the single restriction $x_1 \geq 1$.

At this stage, since the current solution is locally optimal, it is unproductive to proceed with the present line of branching. Thus $x_1 \geq 1$ is replaced by the compulsory branch $x_1 \leq 0$, and the solution process continues.

4. PARAMETRIC B&B AND MIP SENSITIVITY ANALYSIS

Two types of sensitivity analysis are available with parametric B&B as illustrated in the preceding section. The first involves identifying the 'cost' of branches that lead to particular integer solutions. For example, in the earlier numerical example, it was noted that essentially no cost attached to branching on x_2 to obtain the first integer solution. (Consequently, the integer restriction for x_2 was judged 'conditionally superfluous' and the branch was discarded in this instance.) Further, the tableau that yields the integer solution assigns z_1 an objective function coefficient of 1, which means that d_1 can be reduced from its present value of 2 to $2 - 1 = 1$ without altering the LP optimum. This places the cost of the x_1 branch underlying this solution at 1, which provides a measure of the relative significance of this branch and the integer restriction on x_1 in the current solution. Specifically, we can conclude that relaxing this branch will result in at least 1 unit of improvement in the objective function for each unit of change in the value of x_1. Corresponding interpretations apply to situations in which multiple branches underlie an integer solution.

The second type of sensitivity analysis involves the explicit assignment of penalties to deviating from particular problem constraints. Coupling this penalty procedure with parametric B&B, the interactive effect of constraint deviations and parametric branches becomes susceptible to evaluation. For example, the updated cost of a variable representing a constraint violation identifies the net effective cost of imposing the constraint relative to a given integer solution. Thus, if the constraint is in fact violated at this solution, the d_j values for the parametric branches not only reflect branching costs relative to the 'original' objective function but also relative to the penalty incurred from the violated constraint. This enables a more realistic form of problem solving in situations where constraint deviations may be tolerable at a cost. The entire analysis and solution process can be conveniently carried out by merging the compact basis procedures illustrated for the MIP problem with the more elaborate procedures of [7] to maintain the working tableau the same size as the original.

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