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Discrete Optimization 1 (2004) 87–98

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## A very large-scale neighborhood search algorithm for the multi-resource generalized assignment problem

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Received 20 December 2003; received in revised form 26 March 2004; accepted 29 March 2004



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## Abstract

We propose a metaheuristic algorithm for the multi-resource generalized assignment problem (MRGAP). MRGAP is a generalization of the generalized assignment problem, which is one of the representative combinatorial optimization problems known to be NP-hard. The algorithm features a very large-scale neighborhood search, which is a mechanism of conducting the search with complex and powerful moves, where the resulting neighborhood is efficiently searched via the improvement graph. We also incorporate an adaptive mechanism for adjusting search parameters, to maintain a balance between visits to feasible and infeasible regions. Computational comparisons on benchmark instances show that the method is effective, especially for types D and E instances, which are known to be quite difficult.

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**Keywords:** Multi-resource generalized assignment problem; Ejection chain; Very large-scale neighborhood search; Adaptive parameter adjustment; Strategic oscillation

## 1. Introduction

We propose a metaheuristic algorithm for the *multi-resource generalized assignment problem* (MRGAP) [13]. MRGAP seeks a minimum cost assignment of  $n$  jobs to  $m$  agents subject to multi-resource constraints for each agent. MRGAP is a generalization of the generalized assignment problem (GAP) [23, 42, 43] which is one of the representative combinatorial optimization problems known to be NP-hard [37].

Many exact and heuristic algorithms have been proposed for GAP. Among recent exact algorithms for GAP are a branch-and-price algorithm by Savelsbergh [38] and a branch-and-cut algorithm by Naus [23], where exact optimal solutions to many benchmark instances with up to 200 jobs and 20 agents were obtained by Naus [23]. Among various heuristic and metaheuristic algorithms developed for GAP are a combination of the greedy method and local search by Martello and Toth [20, 21]; a tabu search and simulated annealing approach by Osman [26]; a genetic algorithm by Chu and Beasley [7]; variable depth search methods by Amini and Racer [4, 31]; a tabu search approach by Laguna et al. [16]; a set partitioning heuristic by Cattrysse et al. [6]; a relaxation heuristic by Lorena and Narciso [18]; a GRASP and MAX-MIN ant system combined with local search and tabu search by Lourenço and Serra [19]; a linear relaxation heuristic by Trick [41]; a tabu search algorithm by Díaz and Fernández [8]; an ejection chain approach and a path relinking approach by Yagiura et al. [42, 43]; and so on. Motivated by practical applications, various generalizations of GAP have

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been proposed, e.g., the multi-level generalized assignment problem by Laguna et al. [16]; the dynamic multi-resource generalized assignment problem by Shtub and Kogan [39]; the generalized multi-assignment problem by Park et al. [27]; the multi-resource generalized assignment problem with additional constraints by Privault and Herault [30]; and so forth. MRGAP is a natural generalization of GAP, and has many practical applications, e.g., in distributed computer systems and in the trucking industry [11, 12, 22, 29]. For MRGAP, Gavish and Pirkul proposed a branch-and-bound algorithm and two simple Lagrangian heuristics [13]. To the best of our knowledge, however, not much has been done for MRGAP after the work of Gavish and Pirkul in spite of its practical importance.

Our algorithm is based on tabu search, and features a *very large-scale neighborhood search*, which is a mechanism of conducting the search with complex and powerful moves, where the resulting neighborhood is efficiently searched via the *improvement graph* [1, 3, 14]. The idea of the very large-scale neighborhood search is based on the concept of the *ejection chains* [14]. Ejection chains generalize the alternating path constructions of graph theory [5, 9] and also generalize the well-known Lin and Kernighan algorithms [15, 17], which were successfully applied to graph partitioning and traveling salesman problems. Applications of the ejection chain approach include [16, 28, 32, 33, 36]. Recent developments have provided especially effective ejection chain approaches both for the traveling salesman problem [10, 35] and for the vehicle routing problem [34]. We also incorporate the strategic oscillation component of tabu search, which is realized by an automatic mechanism for adjusting search parameters, to maintain a balance between visits to feasible and infeasible regions.

We conducted computational experiments on benchmark instances called types C–E, and compared the proposed method with other existing algorithms. The results show that our algorithm is effective, especially for types D and E instances, which are known to be quite difficult.

## 2. Multi-resource generalized assignment problem

Given  $n$  jobs  $J = \{1, 2, \dots, n\}$  and  $m$  agents  $I = \{1, 2, \dots, m\}$ , we undertake to determine a minimum cost assignment subject to assigning each job to exactly one agent and satisfying multi-resource constraints for each agent, where  $s$  resources  $K = \{1, 2, \dots, s\}$  are considered. Assigning job  $j$  to agent  $i$  incurs a cost of  $c_{ij}$  and consumes an amount  $a_{ijk}$  of each resource  $k \in K$ , whereas the total amount of the resource  $k$  available at agent  $i$  is  $b_{ik}$ . Throughout the paper, we assume  $a_{ijk} \geq 0$  and  $b_{ik} > 0$  for all  $i \in I$ ,  $j \in J$  and  $k \in K$ . An assignment is a mapping  $\sigma : J \rightarrow I$ , where  $\sigma(j) = i$  means that job  $j$  is assigned to agent  $i$ . Let

$$J_i^\sigma = \{j \in J \mid \sigma(j) = i\}, \quad \forall i \in I,$$

which is the set of jobs assigned to agent  $i$  in assignment  $\sigma$ . Then the *multi-resource generalized assignment problem* (MRGAP) is formulated as follows:

$$\begin{aligned} & \text{minimize} && \text{cost}(\sigma) = \sum_{j \in J} c_{\sigma(j), j} \\ & \text{subject to} && \sum_{j \in J_i^\sigma} a_{ijk} \leq b_{ik}, \quad \forall i \in I \text{ and } \forall k \in K. \end{aligned} \tag{1}$$

MRGAP is known to be NP-hard (e.g. [37]), and the (supposedly) simpler problem of judging the existence of a feasible solution for GAP (i.e., MRGAP with  $s = 1$ ) is NP-complete, since the partition problem can be reduced to MRGAP with  $m = 2$  and  $s = 1$ .

## 3. Algorithm

### 3.1. Outline of the algorithm

Our algorithm, called TS-CS (tabu search with chained shift neighborhood), is an extension of local search. Local search starts from an initial solution  $\sigma$  and repeatedly replaces  $\sigma$  with a better solution in its *neighborhood*  $N(\sigma)$  until no better solution is found in  $N(\sigma)$ . The resulting solution  $\sigma$  is *locally optimal* in the sense that no better solution exists in its neighborhood. *Shift* and *swap* neighborhoods  $N_{\text{shift}}$  and  $N_{\text{swap}}$  are usually used in local search methods for GAP, where

$$N_{\text{shift}}(\sigma) = \{\sigma' \mid \sigma' \text{ is obtained from } \sigma \text{ by changing the assignment of one job}\},$$

$$N_{\text{swap}}(\sigma) = \{\sigma' \mid \sigma' \text{ is obtained from } \sigma \text{ by exchanging the assignments of two jobs}\}.$$

The size of the shift neighborhood is  $O(mn)$  and that of the swap neighborhood is  $O(n^2)$ . In addition to these standard neighborhoods, our algorithm uses a *chained shift* neighborhood, which consists of solutions obtainable by certain

sequences of shift moves. The chained shift neighborhood  $N_{\text{chain}}(\sigma)$  is the set of solutions  $\sigma'$  obtainable from  $\sigma$  by shifting  $l$  ( $l = 2, 3, \dots, n$ ) jobs  $j_1, j_2, \dots, j_l$  simultaneously, in such a way that satisfies

$$\begin{aligned} \sigma'(j_r) &= \sigma(j_{r-1}), \quad r = 2, 3, \dots, l, \\ \sigma'(j_1) &= \sigma(j_l). \end{aligned}$$

In other words, for  $r = 2, 3, \dots, l$ , job  $j_r$  is shifted from agent  $\sigma(j_r)$  to agent  $\sigma(j_{r-1})$  after ejecting job  $j_{r-1}$ , and then the cycle is closed by assigning job  $j_1$  to agent  $\sigma(j_l)$ . The length of a chained shift move is the number of jobs  $l$  shifted in the move. This is based on the idea of ejection chains by Glover [14]. Since the size of such a neighborhood can become exponential in  $l$ , we carefully limit its size by utilizing improvement graphs [1, 3]. Since  $|N_{\text{shift}}| \leq |N_{\text{swap}}| \leq |N_{\text{chain}}|$  holds,  $N_{\text{swap}}$  is searched only if  $N_{\text{shift}}$  does not contain an improving solution, and  $N_{\text{chain}}$  is searched only if  $N_{\text{shift}} \cup N_{\text{swap}}$  does not contain an improving solution, unless otherwise stated.

When the search visits the infeasible region, we evaluate the solutions by an objective function penalized by infeasibility:

$$pcost(\sigma) = cost(\sigma) + \sum_{\substack{i \in I \\ k \in K}} \alpha_{ik} p_{ik}(J_i^\sigma), \tag{2}$$

where

$$p_{ik}(S) = \max \left\{ 0, \sum_{j \in S} a_{ijk} - b_{ik} \right\}$$

for  $i \in I, k \in K$  and a subset  $S \subseteq J$  of the jobs. The parameters  $\alpha_{ik}$  ( $> 0$ ) are adaptively controlled during the search by using an algorithm similar to the method in [42].

Whenever the local search stops at a locally optimal solution  $\sigma_{\text{lopt}}$ , it resumes from an initial solution generated by the following rule. We retain a solution  $\sigma_{\text{seed}}$ , which is initially generated randomly, and is replaced with  $\sigma_{\text{lopt}}$  if  $pcost(\sigma_{\text{lopt}}) \leq pcost(\sigma_{\text{seed}})$  holds (the most recent values for  $\alpha_{ik}$  are used in  $pcost$ ) during the search. Then we choose as the initial solution the solution in  $N_{\text{shift}}(\sigma_{\text{seed}}) \setminus T$  with the smallest  $pcost$ , where  $T$  is the set of solutions already generated with shift moves from the current  $\sigma_{\text{seed}}$ . Then the local search starts from the initial solution with neighborhood  $N_{\text{swap}}$  and  $N_{\text{chain}}$ , i.e., the search in  $N_{\text{shift}}$  is forbidden until an improved solution is found. This strategy is confirmed to be effective to avoid cycling of a short period [42].

We will describe the details of the proposed algorithm in the following sections. Section 3.2 explains the improvement graph, Section 3.3 describes how we search an improved solution in the chained shift neighborhood using the improvement graph, Section 3.4 is about the adaptive control mechanism of the penalty weights, and Section 3.5 gives the whole framework of the proposed algorithm.

### 3.2. Improvement graph

The improvement graph  $G(\sigma) = (V, E)$  is a directed graph with a weight  $w_{j_1 j_2}$  on each arc  $(j_1, j_2) \in E$ , where each vertex  $j$  in  $V$  corresponds to a job  $j$  in  $J$  (i.e.,  $V = J$ ), and

$$E = \{(j_1, j_2) \mid j_1, j_2 \in V, \sigma(j_1) \neq \sigma(j_2)\}. \tag{3}$$

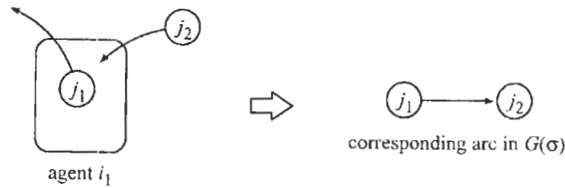
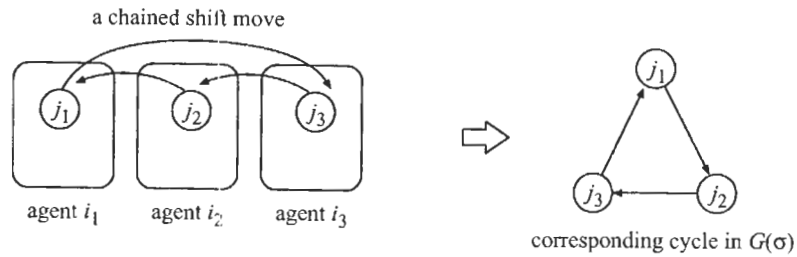
An arc  $(j_1, j_2) \in E$  corresponds to two partial shift moves, ejection of job  $j_1$  from agent  $\sigma(j_1)$  and insertion of job  $j_2$  into  $\sigma(j_1)$  (note that the arc orientation is opposite to the actual move of job  $j_2$ ; this convention is more appropriate to indicate the order of participating actions). A chained shift move of jobs  $j_1, j_2, \dots, j_l$  corresponds to a cycle  $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_l \rightarrow j_1$  in graph  $G(\sigma)$ . The length of a cycle is the number of arcs in it, which is the length of the corresponding chained shift move. We denote

$$i_r = \sigma(j_r), \quad r = 1, 2, \dots, l$$

and assume  $j_{l+1} = j_1$  and  $i_{l+1} = i_1$  for convenience. Fig. 1 illustrates two partial shift moves of jobs  $j_1$  and  $j_2$  and the corresponding arc in the improvement graph, while Fig. 2 illustrates a chained shift move of length three and the corresponding cycle in the improvement graph, where a round rectangle represents an agent and a circle represents a job in the left part of the figures.

For convenience, we denote by

$$pcost_t(S) = \sum_{j \in S} c_{ij} + \sum_{k \in K} \alpha_{ik} p_{ik}(S)$$

Fig. 1. Two partial shift moves and the corresponding arc in  $G(\sigma)$ .Fig. 2. A chained shift move of length three and the corresponding cycle in  $G(\sigma)$ .

the penalized cost for agent  $i \in I$  when the jobs in  $S \subseteq J$  are assigned to  $i$ . Then the penalized cost of (2) can be computed as

$$pcost(\sigma) = \sum_{i \in I} pcost_i(J_i^\sigma).$$

The weight of an arc  $(j_1, j_2)$  is then defined as

$$w_{j_1 j_2} = pcost_{i_1}(J_{i_1}^\sigma \cup \{j_2\} \setminus \{j_1\}) - pcost_{i_1}(J_{i_1}^\sigma),$$

which is the change in  $pcost_{i_1}$  when job  $j_1$  is ejected from agent  $i_1$  and job  $j_2$  is inserted into  $i_1$ .

Let us consider a chained shift move of jobs  $j_1, j_2, \dots, j_l$  and the corresponding cycle  $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_l \rightarrow j_1$  in graph  $G(\sigma)$ . We call a cycle *subset-disjoint* if  $i_r \neq i_{r'}$  holds for any  $r$  and  $r'$  ( $r \neq r' \in \{1, 2, \dots, l\}$ ). Then, if the cycle is subset-disjoint, the change in  $pcost$  by the chained shift move is computed as

$$\begin{aligned} & \sum_{r=1}^l pcost_{i_r}(J_{i_r}^\sigma \cup \{j_{r+1}\} \setminus \{j_r\}) - \sum_{r=1}^l pcost_{i_r}(J_{i_r}^\sigma) \\ &= \sum_{r=1}^l \{pcost_{i_r}(J_{i_r}^\sigma \cup \{j_{r+1}\} \setminus \{j_r\}) - pcost_{i_r}(J_{i_r}^\sigma)\} \\ &= \sum_{r=1}^l w_{j_r j_{r+1}}. \end{aligned}$$

That is, the change in  $pcost$  by a chained shift move is the same as the total weight of arcs in the corresponding cycle. Note that this property may not hold if the cycle is not subset-disjoint. With this property, we can find an improved solution in the chained shift neighborhood by searching for a subset-disjoint cycle with a negative total weight. Unfortunately, the problem of finding such a cycle in an improvement graph is known to be NP-hard in general [40], while the problem of finding a negative cycle, not necessarily subset-disjoint, is known to be polynomially solvable (e.g. [2]). In the case of the traveling salesman problem a special strategy has been proposed that identifies subgraphs of the improvement graph, whose cycles encompass exponential numbers of solutions, over which the problem of finding a negative cycle can be solved in polynomial time [14]. By contrast, in the present context, we allow consideration of the entire graph, but use a tailored heuristic method to find an improved solution in the chained shift neighborhood.

Since the efficiency of the search in the improvement graph crucially depends on the number of arcs, we reduce the number of arcs by the following rule. Let

$$\delta_{j_1 j_2} = |\{k \in K \mid p_{i_1 k}(J_{i_1}^\sigma \setminus \{j_1\} \cup \{j_2\}) > 0\}|,$$

which is the number of overloaded resources at agent  $i_1 = \sigma(j_1)$  when job  $j_1$  is ejected from  $i_1$  and  $j_2$  is inserted into  $i_1$ . We then restrict the arcs to

$$\tilde{E} = \{(j_1, j_2) \in E \mid \delta_{j_1 j_2} = \min_{j' \in J} \delta_{j_1 j'}\}. \tag{4}$$

That is, outgoing arcs from a vertex  $j_1$  is restricted to those whose number of overloaded resources is the minimum among the outgoing arcs from  $j_1$ . With this restriction, we may lose some improved solution in the chained shift neighborhood, but the search will become more efficient. Moreover, the search is biased toward the feasible region. In particular, if  $\delta_{j_r, j_{r+1}} = 0$  holds for all  $r = 1, 2, \dots, l$  in a subset-disjoint cycle  $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_l \rightarrow j_1$ , then the solution obtained by applying the corresponding chained shift move is feasible.

### 3.3. Search in the chained shift neighborhood

In this section, we explain the algorithm to find an improved solution in the chained shift neighborhood via the improvement graph  $\tilde{G}(\sigma) = (V, \tilde{E})$ . We do not restrict our search to subset-disjoint cycles, since we expect that the total arc weight of a cycle is often a good estimate on the change in *pcost* of the corresponding chained shift move even if the cycle is not subset-disjoint. Note that a cycle of length at most three is always subset-disjoint, which is immediate from definition (3). Moreover, as shown in Appendix, a short cycle of length at most  $\sqrt{m}$  tends to be subset-disjoint if the assignment  $\sigma$  is random. On the other hand, a longer cycle tends not to be subset-disjoint (Appendix), and a cycle of length more than  $m$  cannot be subset-disjoint. We therefore restrict the search to cycles of length at most  $m$ .

Our algorithm is based on dynamic programming. Let  $f^*(j_1, j, l)$  be the minimum total weight of a path among those of length  $l$  from vertex  $j_1$  to  $j$  for  $j_1, j \in J, l = 1, 2, \dots, m - 1$ . We assume  $w_{j_1 j_2} = +\infty$  for  $(j_1, j_2) \notin \tilde{E}$  throughout this section. Then  $f^*(j_1, j, l)$  can be computed by

$$f^*(j_1, j, l) = \begin{cases} w_{j_1 j}, & l = 1, \\ \min_{j' \in J} \{f^*(j_1, j', l - 1) + w_{j' j}\}, & l = 2, 3, \dots, m - 1 \end{cases} \tag{5}$$

for  $j_1, j \in J$ . During the recursion, we retain a  $j'$  that attains the minimum in (5), for each  $(j_1, j, l)$ . Then a path of weight  $f^*(j_1, j, l)$  can be reconstructed by tracing such  $j'$  from  $l$  to 1.

By definition,  $f^*(j_1, j, l) + w_{j j_1}$  represents the minimum weight among those cycles of length  $l$  that include edge  $(j, j_1)$ . We therefore can find a cycle of negative weight in the improvement graph by searching  $(j_1, j, l)$  with  $f^*(j_1, j, l) + w_{j j_1} < 0$ . While searching such  $(j_1, j, l)$ , we recompute *pcost* of the corresponding solution whenever we find  $(j_1, j, l)$  that satisfies  $f^*(j_1, j, l) + w_{j j_1} < 0$ , since  $f^*(j_1, j, l) + w_{j j_1}$  may not be the same as the change in *pcost* if the cycle is not subset-disjoint. This procedure is summarized as follows, which is called *Search-Chained-Shift (SCS)*.

#### Procedure SCS

*Input:* the current solution  $\sigma$ .

*Output:* ‘no’ if it failed in finding an improved solution in  $N_{\text{chain}}(\sigma)$ ; otherwise an improved solution  $\sigma' \in N_{\text{chain}}(\sigma)$ .

*Step 1:* Let  $S := J$ .

*Step 2:* If  $S = \emptyset$  holds, output ‘no’ and stop. Otherwise, choose a  $j_1 \in S$  randomly, and let  $S := S \setminus \{j_1\}$  and  $l := 1$ .

*Step 3:* Compute  $f^*(j_1, j, l)$  for all  $j$  by (5).

*Step 4:* Sort  $j \in J$  in nondecreasing order of  $f^*(j_1, j, l) + w_{j j_1}$ . Then, for each  $j \in J$  in this order, compute the change in *pcost* of the corresponding chained shift move, and if an improved solution  $\sigma$  is found, output  $\sigma$  immediately and stop.

*Step 5:* If  $l \leq m - 2$ , let  $l := l + 1$  and return to Step 3; otherwise return to Step 2.

Procedure SCS can find a cycle of negative weight (not necessarily subset-disjoint) if it exists; however, it does not necessarily mean that an improved solution is found in the chained shift neighborhood, since the weight of a cycle may not be the same as the change in *pcost* by the corresponding chained shift move.

The recursion of (5) is computed as in Fig. 3. The time complexity for computing  $f^*(j_1, j, l)$  for all  $j_1, j \in J$  and  $l = 1, 2, \dots, m - 1$  is  $O(nm(n + |\tilde{E}|)) = O(nm|\tilde{E}|)$ , which is the dominant part of the time complexity of procedure SCS. More precisely,  $O(n \log n)$  time is necessary for sorting for each call to Step 4, which requires  $O(n^2 m \log n)$  time in total. Moreover,  $O(ls)$  time is necessary to recompute *pcost* for each  $(j_1, j, l)$  with  $f^*(j_1, j, l) + w_{j j_1} < 0$ , where  $s$  is the number of resources. This may require  $O(n^2 m^2 s)$  time in total, since the number of such  $(j_1, j, l)$  is  $O(n^2 m)$  and  $l \leq m$ . Hence the worst case time complexity is  $O(nm(|\tilde{E}| + n \log n + nms))$ . However, the latter two parts in this complexity usually do not dominate  $O(nm|\tilde{E}|)$ , because (1) the improvement graph is usually dense even with the restriction of (4), and (2) the number of combinations  $(j_1, j, l)$  with  $f^*(j_1, j, l) + w_{j j_1} < 0$  is usually very small.

```

for  $j_1 = 1, 2, \dots, n$  do
   $f^*(j_1, j, 1) := w_{j_1 j}$  for all  $j \in J$ .
  for  $l = 2, 3, \dots, m - 1$  do
     $f^*(j_1, j, l) := +\infty$  for all  $j \in J$ .
    for each  $(j', j) \in \tilde{E}$  do
       $f^*(j_1, j, l) := f^*(j_1, j', l - 1) + w_{j' j}$  if  $f^*(j_1, j', l - 1) + w_{j' j} < f^*(j_1, j, l)$ .
    end for
  end for
end for.

```

Fig. 3. Algorithm to compute the recursion of (5).

In order to make the search more efficient, we employ the following restricted dynamic programming. Let

$$\lambda(x) = \begin{cases} +\infty, & x \geq 0, \\ x, & x < 0, \end{cases}$$

then the recursion of the restricted dynamic programming is

$$f^*(j_1, j, l) = \begin{cases} \lambda(w_{j_1 j}), & l = 1, \\ \min_{j' \in J} \{ \lambda(f^*(j_1, j', l - 1) + w_{j' j}) \}, & l = 2, 3, \dots, m - 1 \end{cases} \quad (6)$$

for  $j_1, j \in J$ . Compared to (5), the recursion of (6) restricts the search to only those paths  $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_l$  in which  $f^*(j_1, j_{l'}, l' - 1)$  is negative for all  $l' = 2, 3, \dots, l$ . The recursion of (6) is usually more efficient than (5), because we can omit the computation if  $f^*(j_1, j', l - 1) + w_{j' j} \geq 0$  holds. To realize this, we sort the outgoing edges of each vertex in nondecreasing order of the weight, and scan the outgoing edges  $(j', j)$  from a vertex  $j'$  in this order until  $f^*(j_1, j', l - 1) + w_{j' j} \geq 0$  holds. The sorting is necessary only once for each vertex for a given  $\tilde{G}(\sigma)$ . This takes  $O(n^2 \log n)$  time in total, which is negligible compared to the time needed for the DP recursion. Procedure SCS with this restricted dynamic programming can also find a cycle of negative weight (not necessarily subset-disjoint) if it exists. This fact is immediate from the following well-known lemma (see, e.g., [17]).

**Lemma 1.** *If  $\sum_{r=1}^l w_{j_r j_{r+1}} < 0$  holds, then there exists a  $u$  such that  $\sum_{r=u}^l w_{j_r j_{r+1}} < 0$  holds for all  $t = u, u + 1, \dots, l$ , and  $\sum_{r=1}^t w_{j_r j_{r+1}} + \sum_{r=u}^l w_{j_r j_{r+1}} < 0$  holds for all  $t = 1, 2, \dots, u - 1$ .*

**Proof.** Let  $t^*$  be the largest index among those  $t$  that maximizes  $\sum_{r=1}^t w_{j_r j_{r+1}}$ . Then  $\sum_{r=1}^t w_{j_r j_{r+1}} + \sum_{r=t^*+1}^l w_{j_r j_{r+1}} \leq \sum_{r=1}^{t^*} w_{j_r j_{r+1}} + \sum_{r=t^*+1}^l w_{j_r j_{r+1}} < 0$  holds if  $1 \leq t \leq t^*$ , and  $\sum_{r=t^*+1}^l w_{j_r j_{r+1}} = \sum_{r=1}^l w_{j_r j_{r+1}} - \sum_{r=1}^{t^*} w_{j_r j_{r+1}} < 0$  holds if  $t^* + 1 \leq t \leq l$ . Hence  $u = t^* + 1$  satisfies the conditions in the lemma.  $\square$

### 3.4. Adaptive control of the penalty weights

In this section, we explain the adaptive mechanism for controlling the penalty weights  $\alpha_{ik}$  in algorithm TS-CS. The performance of TS-CS highly depends on the values of  $\alpha_{ik}$ .

Initially we set  $\alpha_{ik} := \beta$  for all  $i$  and  $k$ , where we use  $\beta = 1$  in the computational experiment in Section 4. We then update  $\alpha_{ik}$  by the following rule whenever a locally optimal solution  $\sigma_{\text{lopt}}$  with respect to the current  $\alpha_{ik}$  is reached. Here we use functions

$$q_{ik}^{\text{inc}}(\sigma_{\text{lopt}}) = p_{ik}(\sigma_{\text{lopt}})/b_{ik},$$

$$q_{ik}^{\text{dec}}(\sigma_{\text{lopt}}) = \begin{cases} -1, & \text{if } p_{ik}(\sigma_{\text{lopt}}) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The rule depends on whether a feasible solution was found or not during the search after the last update of  $\alpha_{ik}$ .

Case 1 (If no feasible solution was found after the last update of  $\alpha_{ik}$ ): The  $\alpha_{ik}$  values are increased for all  $i \in I$  and  $k \in K$  by

$$\alpha_{ik} := \alpha_{ik}(1 + \Delta_{ik}),$$

where

$$\Delta_{ik} = \begin{cases} \text{step\_size\_inc} \cdot \frac{q_{ik}^{\text{inc}}(\sigma_{\text{lopt}})}{\max_{i' \in I, k' \in K} |q_{i'k'}^{\text{inc}}(\sigma_{\text{lopt}})|}, & \text{if } \max_{i' \in I, k' \in K} |q_{i'k'}^{\text{inc}}(\sigma_{\text{lopt}})| > 0, \\ 0, & \text{otherwise} \end{cases}$$

(*step\\_size\\_inc* > 0 is a prespecified parameter).

Case 2 (Otherwise): All  $\alpha_{ik}$  are decreased. The rule to update  $\alpha_{ik}$  is the same as Case 1 except that  $q_{ik}^{\text{dec}}(\sigma_{\text{lopt}})$  and *step\\_size\\_dec* (a prespecified parameter satisfying  $0 < \text{step\_size\_dec} < 1$ ) are used instead of  $q_{ik}^{\text{inc}}(\sigma_{\text{lopt}})$  and *step\\_size\\_inc*.

The meaning of these rules is explained as follows. In the functions  $q_{ik}^{\text{inc}}(\sigma_{\text{lopt}})$ , the penalties  $p_{ik}(\sigma_{\text{lopt}})$  are divided by  $b_{ik}$  to normalize among agents and resources. If no feasible solution was found after the last update of  $\alpha_{ik}$  (Case 1), we increase  $\alpha_{ik}$  by  $\alpha_{ik}\Delta_{ik}$  for all  $i \in I$  and  $k \in K$ , where the  $q_{ik}^{\text{inc}}(\sigma_{\text{lopt}})$  terms are the weights to emphasize overloaded agents, and  $\Delta_{ik}$  are proportional to  $q_{ik}^{\text{inc}}(\sigma_{\text{lopt}})$  normalized by  $\max_{i' \in I, k' \in K} |q_{i'k'}^{\text{inc}}(\sigma_{\text{lopt}})|$  so that the maximum amount of changes among all agents becomes *step\\_size\\_inc*. The opposite case (Case 2) is similarly explained. In our preliminary experiments, we found that the performance of TS-CS is robust with parameters *step\\_size\\_inc* and *step\\_size\\_dec*. In Section 4, we use *step\\_size\\_inc* = 0.01 and *step\\_size\\_dec* = 0.1.

### 3.5. The whole framework of the algorithm

The whole framework of the proposed algorithm TS-CS is described in this section. The search starts from a random solution. The current solution is improved by local search with the shift, swap and chained shift neighborhoods, where the swap (resp., chained shift) neighborhood is searched only if the current solution is locally optimal with respect to the shift (resp., swap) neighborhood. Whenever the local search stops at a locally optimal solution  $\sigma_{\text{lopt}}$ , the penalty weights  $\alpha_{ik}$  are updated, and then the search resumes from an initial solution generated by applying a shift move to a solution  $\sigma_{\text{seed}}$ , which is initially generated randomly, and is replaced with  $\sigma_{\text{lopt}}$  if  $\text{pcost}(\sigma_{\text{lopt}}) \leq \text{pcost}(\sigma_{\text{seed}})$  holds. Tabu list  $T$  is used to avoid short cycling when we choose a shift move from  $\sigma_{\text{seed}}$ , where  $T$  consists of those solutions to which the moves are forbidden. The search stops when the execution time exceeds *timelim* (a prespecified parameter). We check the feasibility for all the solutions generated during the search, and update the incumbent solution whenever a feasible solution is found.

#### Algorithm TS-CS

Step 1: Generate a solution  $\sigma$  randomly, i.e., for each  $j \in J$ , choose an  $i$  uniformly at random from  $\{1, 2, \dots, m\}$  and let  $\sigma(j) := i$ . Let  $T := \emptyset$  and  $\sigma_{\text{seed}} := \sigma$ .

Step 2: Search the shift neighborhood with the first admissible move strategy. If a solution  $\sigma' \in N_{\text{shift}}(\sigma)$  that satisfies  $\text{pcost}(\sigma') < \text{pcost}(\sigma)$  is found, let  $\sigma := \sigma'$  and return to Step 2.

Step 3: Search the swap neighborhood with the first admissible move strategy. If a solution  $\sigma' \in N_{\text{swap}}(\sigma)$  that satisfies  $\text{pcost}(\sigma') < \text{pcost}(\sigma)$  is found, let  $\sigma := \sigma'$  and return to Step 2.

Step 4: Search the chained shift neighborhood by procedure SCS. If a solution  $\sigma' \in N_{\text{chain}}(\sigma)$  that satisfies  $\text{pcost}(\sigma') < \text{pcost}(\sigma)$  is found, let  $\sigma := \sigma'$  and return to Step 2.

Step 5: If the computation time exceeds *timelim*, output the incumbent solution and stop.

Step 6: Update the penalty weights  $\alpha_{ik}$  by using the procedure in Section 3.4.

Step 7: If  $\text{pcost}(\sigma) \leq \text{pcost}(\sigma_{\text{seed}})$  and  $\sigma \neq \sigma_{\text{seed}}$ , let  $T := \emptyset$  and  $\sigma_{\text{seed}} := \sigma$ ; otherwise let  $\sigma := \sigma_{\text{seed}}$ .

Step 8: Let  $\sigma'$  be the solution that minimizes *pcost* in  $N_{\text{shift}}(\sigma_{\text{seed}}) \setminus T$ . Let  $\sigma := \sigma'$  and  $T := T \cup \{\sigma'\}$ . Return to Step 3.

## 4. Computational results

We compared the proposed algorithm TS-CS with the following three algorithms: (1) tabu search without chained shift neighborhood (denoted TS-noCS), (2) a general problem solver for the weighted constraint satisfaction problem proposed in [24] (denoted TS-WCSP), and (3) a commercial exact solver CPLEX 6.5 (denoted CPLEX).<sup>1</sup> Note that TS-noCS is

<sup>1</sup> We also tested a later version, CPLEX 8.1.0, for some instances, which is licensed on a different computer, but the results are not much different.

Table 1  
Results for type C instances

| <i>n</i> | <i>m</i> | <i>s</i> | LB                | TS-CS             |        | TS-noCS           |        | TS-WCSP           |        | CPLEX             |       |
|----------|----------|----------|-------------------|-------------------|--------|-------------------|--------|-------------------|--------|-------------------|-------|
|          |          |          |                   | Best              | TTB    | Best              | TTB    | Best              | TTB    | Best              | TTB   |
| 100      | 5        | 1        | 1931 <sup>a</sup> | 1931 <sup>b</sup> | 1.25   | 1931 <sup>b</sup> | 0.61   | 1933              | 30.00  | 1931 <sup>b</sup> | 2     |
| 100      | 5        | 2        | 1933 <sup>a</sup> | 1933 <sup>b</sup> | 2.39   | 1933 <sup>b</sup> | 58.54  | 1933 <sup>b</sup> | 15.68  | 1933 <sup>b</sup> | 0     |
| 100      | 5        | 4        | 1943 <sup>a</sup> | 1943 <sup>b</sup> | 172.35 | 1943 <sup>b</sup> | 197.17 | 1944              | 91.65  | 1943 <sup>b</sup> | 19    |
| 100      | 5        | 8        | 1950 <sup>a</sup> | 1950 <sup>b</sup> | 61.19  | 1950 <sup>b</sup> | 185.08 | 1956              | 179.87 | 1950 <sup>b</sup> | 26    |
| 100      | 10       | 1        | 1402 <sup>a</sup> | 1402 <sup>b</sup> | 5.49   | 1402 <sup>b</sup> | 28.92  | 1402 <sup>b</sup> | 15.87  | 1402 <sup>b</sup> | 9     |
| 100      | 10       | 2        | 1409 <sup>a</sup> | 1409 <sup>b</sup> | 133.34 | 1410              | 74.06  | 1411              | 261.89 | 1409 <sup>b</sup> | 35    |
| 100      | 10       | 4        | 1419 <sup>a</sup> | 1419 <sup>b</sup> | 37.40  | 1419 <sup>b</sup> | 53.13  | 1419 <sup>b</sup> | 93.59  | 1419 <sup>b</sup> | 38    |
| 100      | 10       | 8        | 1435 <sup>a</sup> | 1436              | 147.34 | 1440              | 208.26 | 1435 <sup>b</sup> | 227.31 | 1435 <sup>b</sup> | 271   |
| 100      | 20       | 1        | 1243 <sup>a</sup> | 1245              | 55.48  | 1245              | 43.30  | 1245              | 140.22 | 1243 <sup>b</sup> | 8     |
| 100      | 20       | 2        | 1250 <sup>a</sup> | 1251              | 45.39  | 1252              | 46.16  | 1253              | 202.30 | 1250 <sup>b</sup> | 5     |
| 100      | 20       | 4        | 1254 <sup>a</sup> | 1257              | 186.09 | 1256              | 294.35 | 1258              | 135.77 | 1254 <sup>b</sup> | 30    |
| 100      | 20       | 8        | 1267 <sup>a</sup> | 1269              | 205.60 | 1275              | 261.05 | 1267 <sup>b</sup> | 11.86  | 1272              | 199   |
| Average  |          |          | 1536.3            | 1537.1            | 87.78  | 1538.0            | 120.89 | 1538.0            | 117.17 | 1536.8            | 53.5  |
| 200      | 5        | 1        | 3456 <sup>a</sup> | 3456 <sup>b</sup> | 170.10 | 3458              | 5.91   | 3460              | 107.54 | 3456 <sup>b</sup> | 35    |
| 200      | 5        | 2        | 3461 <sup>a</sup> | 3461 <sup>b</sup> | 47.64  | 3461 <sup>b</sup> | 219.14 | 3462              | 180.82 | 3461 <sup>b</sup> | 17    |
| 200      | 5        | 4        | 3466 <sup>a</sup> | 3466 <sup>b</sup> | 167.53 | 3466 <sup>b</sup> | 196.33 | 3469              | 389.58 | 3466 <sup>b</sup> | 173   |
| 200      | 5        | 8        | 3473 <sup>a</sup> | 3473 <sup>b</sup> | 530.30 | 3473 <sup>b</sup> | 447.35 | 3478              | 294.82 | 3474              | 56    |
| 200      | 10       | 1        | 2806 <sup>a</sup> | 2807              | 46.23  | 2808              | 156.07 | 2811              | 53.63  | 2806 <sup>b</sup> | 249   |
| 200      | 10       | 2        | 2811 <sup>a</sup> | 2812 <sup>b</sup> | 316.68 | 2813              | 167.54 | 2812 <sup>b</sup> | 352.65 | 2812 <sup>b</sup> | 37    |
| 200      | 10       | 4        | 2819 <sup>a</sup> | 2821              | 399.99 | 2821              | 444.46 | 2823              | 151.04 | 2819 <sup>b</sup> | 347   |
| 200      | 10       | 8        | 2833              | 2837 <sup>b</sup> | 344.24 | 2842              | 381.29 | 2842              | 480.93 | 2842              | 134   |
| 200      | 20       | 1        | 2391 <sup>a</sup> | 2393              | 369.54 | 2399              | 136.93 | 2394              | 31.13  | 2391 <sup>b</sup> | 296   |
| 200      | 20       | 2        | 2397 <sup>a</sup> | 2398 <sup>b</sup> | 313.10 | 2400              | 155.53 | 2403              | 348.75 | 2398 <sup>b</sup> | 175   |
| 200      | 20       | 4        | 2408              | 2409 <sup>b</sup> | 430.56 | 2416              | 165.83 | 2415              | 15.17  | 2415              | 115   |
| 200      | 20       | 8        | 2415              | 2422              | 47.29  | 2424              | 419.02 | 2423              | 176.03 | 2419 <sup>b</sup> | 451   |
| Average  |          |          | 2894.7            | 2896.3            | 265.27 | 2898.4            | 241.28 | 2899.3            | 215.17 | 2896.6            | 173.8 |

<sup>a</sup>Indicates that the value is optimal.

<sup>b</sup>Indicates that the best cost among the tested algorithms is attained.

the same as TS-CS except that it does not use the chained shift neighborhood. All the algorithms were coded in C and run on a workstation Sun Ultra 2 Model 2300 (two UltraSPARC II 300MHz processors with 1 GB memory), where the computation was executed on a single processor.

Test instances were generated randomly by using benchmark instances for GAP. (We use GAP instances for  $s = 1$ .) There are five types of benchmark instances of GAP called types A–E [7, 16]. Out of these, we use three types C–E, since the other two are too easy to see differences among the tested algorithms. Instances of these types are generated as follows:

Type C:  $a_{ij1}$  are random integers from [5, 25],  $c_{ij}$  are random integers from [10, 50], and  $b_{i1} = 0.8 \sum_{j \in J} a_{ij1}/m$ .

Type D:  $a_{ij1}$  are random integers from [1, 100],  $c_{ij} = 111 - a_{ij1} + e_{ij}$ , where  $e_{ij}$  are random integers from  $[-10, 10]$ , and  $b_{i1} = 0.8 \sum_{j \in J} a_{ij1}/m$ .

Type E:  $a_{ij1} = 1 - 10 \ln e_{ij}$ , where  $e_{ij}$  are random numbers from (0, 1],  $c_{ij} = 1000/a_{ij1} - 10\hat{e}_{ij}$ , where  $\hat{e}_{ij}$  are random numbers from [0, 1], and  $b_{i1} = 0.8 \sum_{j \in J} a_{ij1}/m$ .

Types D and E are somewhat harder than type C, since  $c_{ij}$  and  $a_{ij1}$  are inversely correlated. We tested 18 instances of types C–E with  $n$  up to 200. Among them, types C and D instances were taken from OR-Library,<sup>2</sup> and type E instances were generated by us, and are available at our web site.<sup>3</sup> For each of these GAP instances, we generated MRGAP instances by setting  $a_{ijk} = 3a_{ij1}/4 + \gamma_{ijk}a_{ij1}/2$  for each  $k = 2, 3, \dots, s$  as in [13], where  $\gamma_{ijk}$  are random numbers from [0, 1], and then setting  $b_k$  for  $k = 2, 3, \dots, s$  in the same manner as  $b_{i1}$  according to the problem type. As a result, there

<sup>2</sup> URL of OR-Library: <http://mscmga.ms.ic.ac.uk/jeb/orlib/gapinfo.html>.

<sup>3</sup> URL of our web site for GAP instances: <http://www-or.amp.i.kyoto-u.ac.jp/~yagiura/gap/>.

Table 2  
Results for type D instances

| $n$     | $m$ | $s$ | LB                | TS-CS              |        | TS-noCS           |        | TS-WCSP |        | CPLEX              |       |
|---------|-----|-----|-------------------|--------------------|--------|-------------------|--------|---------|--------|--------------------|-------|
|         |     |     |                   | Best               | TTB    | Best              | TTB    | Best    | TTB    | Best               | TTB   |
| 100     | 5   | 1   | 6353 <sup>a</sup> | 6357 <sup>b</sup>  | 109.87 | 6359              | 209.92 | 6370    | 115.87 | 6358               | 43    |
| 100     | 5   | 2   | 6352              | 6359 <sup>b</sup>  | 136.10 | 6371              | 53.42  | 6380    | 106.03 | 6360               | 27    |
| 100     | 5   | 4   | 6362              | 6379 <sup>b</sup>  | 207.25 | 6381              | 179.95 | 6404    | 297.04 | 6386               | 172   |
| 100     | 5   | 8   | 6388              | 6425 <sup>b</sup>  | 67.89  | 6428              | 239.62 | 6500    | 264.45 | 6428               | 244   |
| 100     | 10  | 1   | 6342              | 6361 <sup>b</sup>  | 246.00 | 6377              | 79.09  | 6418    | 192.77 | 6381               | 132   |
| 100     | 10  | 2   | 6340              | 6378 <sup>b</sup>  | 174.39 | 6405              | 168.45 | 6411    | 241.94 | 6419               | 88    |
| 100     | 10  | 4   | 6361              | 6430 <sup>b</sup>  | 274.92 | 6438              | 184.59 | 6516    | 126.63 | 6468               | 166   |
| 100     | 10  | 8   | 6388              | 6478 <sup>b</sup>  | 241.80 | 6520              | 232.30 | 6679    | 255.64 | 6528               | 83    |
| 100     | 20  | 1   | 6177              | 6231 <sup>b</sup>  | 194.94 | 6270              | 217.92 | 6305    | 204.90 | 6280               | 60    |
| 100     | 20  | 2   | 6165              | 6261 <sup>b</sup>  | 253.83 | 6305              | 56.77  | 6389    | 223.84 | 6316               | 19    |
| 100     | 20  | 4   | 6182              | 6321 <sup>b</sup>  | 277.59 | 6331              | 178.93 | 6529    | 58.70  | 6406               | 148   |
| 100     | 20  | 8   | 6206              | 6482               | 234.49 | 6481 <sup>b</sup> | 270.78 | 6736    | 34.08  | 6588               | 68    |
| Average |     |     | 6301.3            | 6371.8             | 201.59 | 6388.8            | 172.65 | 6469.8  | 176.82 | 6409.8             | 104.2 |
| 200     | 5   | 1   | 12741             | 12751              | 191.63 | 12756             | 81.33  | 12760   | 87.70  | 12750 <sup>b</sup> | 62    |
| 200     | 5   | 2   | 12751             | 12766 <sup>b</sup> | 441.53 | 12772             | 110.91 | 12778   | 171.48 | 12766 <sup>b</sup> | 534   |
| 200     | 5   | 4   | 12745             | 12775              | 178.92 | 12778             | 151.80 | 12799   | 78.63  | 12762 <sup>b</sup> | 286   |
| 200     | 5   | 8   | 12755             | 12805              | 527.78 | 12809             | 292.53 | 12844   | 348.73 | 12787 <sup>b</sup> | 432   |
| 200     | 10  | 1   | 12426             | 12463              | 330.52 | 12482             | 555.38 | 12478   | 279.46 | 12457 <sup>b</sup> | 27    |
| 200     | 10  | 2   | 12431             | 12477 <sup>b</sup> | 476.07 | 12518             | 512.70 | 12533   | 590.21 | 12482              | 578   |
| 200     | 10  | 4   | 12432             | 12496 <sup>b</sup> | 471.01 | 12552             | 540.32 | 12586   | 548.07 | 12532              | 112   |
| 200     | 10  | 8   | 12448             | 12571 <sup>b</sup> | 481.10 | 12592             | 513.42 | 12812   | 346.33 | 12577              | 209   |
| 200     | 20  | 1   | 12230             | 12312 <sup>b</sup> | 230.72 | 12365             | 346.77 | 12409   | 445.56 | 12393              | 297   |
| 200     | 20  | 2   | 12227             | 12332 <sup>b</sup> | 597.11 | 12384             | 568.76 | 12442   | 587.38 | 12425              | 21    |
| 200     | 20  | 4   | 12237             | 12396 <sup>b</sup> | 576.45 | 12488             | 386.71 | 12605   | 573.62 | 12472              | 425   |
| 200     | 20  | 8   | 12254             | 12485 <sup>b</sup> | 337.27 | 12650             | 481.50 | 12918   | 250.78 | 12548              | 132   |
| Average |     |     | 12473.1           | 12552.4            | 403.34 | 12595.5           | 378.51 | 12663.7 | 359.00 | 12579.3            | 259.6 |

<sup>a</sup>Means that the value is optimal.

<sup>b</sup>Means that the best cost among the tested algorithms is attained.

are 72 instances, i.e., one instance for each combination of the type (C, D or E),  $n$  (100 or 200),  $m$  (5, 10 or 20) and  $s$  (1, 2, 4 or 8). The generated MRGAP instances are available at our web site.<sup>4</sup>

Tables 1–3 show the results of TS-CS, TS-noCS, TS-WCSP and CPLEX, where the time limit is set to 300 (resp., 600) seconds for each instance with  $n = 100$  (resp., 200). Columns “best” show the objective values of the best solutions obtained by the algorithms within the time limit, and columns “TTB” (time to best) show the CPU seconds when the best solutions were found for the first time. (Precise TTB for CPLEX is not available, and hence an approximate value is shown.) Columns “LB” show the lower bounds on the optimal values, where the mark “Q” indicates that the value is optimal. (Most of these lower bounds were obtained by CPLEX, where the time limit was set to 3600 seconds. Some optimal values for GAP (i.e.,  $s = 1$ ) were found by Nauss [23], and some LBs for GAP were reported in [42], which were found by solving a Lagrangian relaxation problem. If  $s \geq 2$ , then optimal values and LBs were found by CPLEX.) In the tables, each “b” mark represents that the best cost is attained, and “—” means that no feasible solution was found. The average of “LB,” “best” and “TTB” for  $n = 100$  and 200, respectively, are also shown.

From the tables, we can observe the following:

- The performance of algorithm TS-CS is better than TS-noCS, especially for types D and E instances. This indicates that incorporating the chained shift neighborhood is effective for hard instances.
- The performance of TS-CS is better than TS-WCSP and CPLEX. CPLEX is very effective for type C instances; however, TS-CS is better for types D and E instances.

<sup>4</sup> URL of our web site for MRGAP instances: <http://www-or.amp.i.kyoto-u.ac.jp/~yagiura/mrgap/>.

Table 3  
Results for type E instances

| <i>n</i> | <i>m</i> | <i>s</i> | LB                 | TS-CS              |        | TS-noCS            |        | TS-WCSP |        | CPLEX              |       |
|----------|----------|----------|--------------------|--------------------|--------|--------------------|--------|---------|--------|--------------------|-------|
|          |          |          |                    | Best               | TTB    | Best               | TTB    | Best    | TTB    | Best               | TTB   |
| 100      | 5        | 1        | 12681 <sup>a</sup> | 12681 <sup>b</sup> | 54.08  | 12682              | 124.34 | 12753   | 2.72   | 12681 <sup>b</sup> | 44    |
| 100      | 5        | 2        | 12692 <sup>a</sup> | 12692 <sup>b</sup> | 120.93 | 12692 <sup>b</sup> | 210.25 | 12727   | 28.90  | 12692 <sup>b</sup> | 212   |
| 100      | 5        | 4        | 12810 <sup>a</sup> | 12812              | 104.03 | 12812              | 83.04  | 12893   | 79.48  | 12810 <sup>b</sup> | 24    |
| 100      | 5        | 8        | 12738 <sup>a</sup> | 12738 <sup>b</sup> | 53.98  | 12739              | 210.64 | 12876   | 152.79 | 12749              | 61    |
| 100      | 10       | 1        | 11577 <sup>a</sup> | 11577 <sup>b</sup> | 90.03  | 11584              | 91.42  | 11712   | 209.98 | 11584              | 200   |
| 100      | 10       | 2        | 11582 <sup>a</sup> | 11587 <sup>b</sup> | 179.65 | 11604              | 290.33 | 11665   | 41.33  | 11612              | 80    |
| 100      | 10       | 4        | 11636              | 11676 <sup>b</sup> | 289.32 | 11689              | 106.93 | 11864   | 102.15 | 11753              | 257   |
| 100      | 10       | 8        | 11619              | 11701 <sup>b</sup> | 260.90 | 11756              | 230.60 | 11836   | 86.49  | 11739              | 258   |
| 100      | 20       | 1        | 8436 <sup>a</sup>  | 8447 <sup>b</sup>  | 142.39 | 8488               | 110.94 | 8655    | 144.76 | 8565               | 72    |
| 100      | 20       | 2        | 10123              | 10150 <sup>b</sup> | 207.09 | 10219              | 247.15 | 10471   | 79.65  | 10251              | 234   |
| 100      | 20       | 4        | 10794              | 11029 <sup>b</sup> | 160.57 | 11075              | 82.28  | 11271   | 117.97 | 11443              | 78    |
| 100      | 20       | 8        | 11224              | 11610 <sup>b</sup> | 265.33 | 11817              | 285.93 | 11957   | 143.07 | 12458              | 291   |
| Average  |          |          | 11492.7            | 11558.3            | 160.69 | 11596.4            | 172.82 | 11723.3 | 99.11  | 11694.8            | 150.9 |
| 200      | 5        | 1        | 24930 <sup>a</sup> | 24933              | 43.21  | 24933              | 399.77 | 25002   | 101.77 | 24930 <sup>b</sup> | 18    |
| 200      | 5        | 2        | 24933 <sup>a</sup> | 24936              | 430.29 | 24933 <sup>b</sup> | 520.42 | 25024   | 519.47 | 24933 <sup>b</sup> | 131   |
| 200      | 5        | 4        | 24990              | 24999 <sup>b</sup> | 537.27 | 25017              | 555.53 | 25091   | 280.81 | 25003              | 228   |
| 200      | 5        | 8        | 24943 <sup>a</sup> | 24950              | 192.28 | 24970              | 449.00 | 25090   | 271.98 | 24943 <sup>b</sup> | 68    |
| 200      | 10       | 1        | 23307 <sup>a</sup> | 23312 <sup>b</sup> | 411.12 | 23326              | 30.46  | 23414   | 494.72 | 23321              | 140   |
| 200      | 10       | 2        | 23310              | 23317 <sup>b</sup> | 436.01 | 23333              | 51.82  | 23538   | 303.94 | 23325              | 386   |
| 200      | 10       | 4        | 23344              | 23363 <sup>b</sup> | 376.65 | 23412              | 56.24  | 23628   | 83.62  | 23543              | 309   |
| 200      | 10       | 8        | 23339              | 23412              | 198.96 | 23410 <sup>b</sup> | 357.69 | 23714   | 550.29 | 23744              | 56    |
| 200      | 20       | 1        | 22379 <sup>a</sup> | 22386 <sup>b</sup> | 178.16 | 22455              | 519.64 | 22815   | 207.84 | 22457              | 185   |
| 200      | 20       | 2        | 22387              | 22408 <sup>b</sup> | 333.97 | 22459              | 121.13 | 22834   | 377.76 | 22558              | 302   |
| 200      | 20       | 4        | 22395              | 22439 <sup>b</sup> | 462.53 | 22517              | 566.93 | 22990   | 413.64 | 22782              | 238   |
| 200      | 20       | 8        | 22476              | 22614 <sup>b</sup> | 317.52 | —                  | —      | 23057   | 394.42 | 23482              | 85    |
| Average  |          |          | 23561.1            | 23589.1            | 326.50 | —                  | —      | 23849.8 | 333.36 | 23751.8            | 178.8 |

<sup>a</sup>Indicates that the value is optimal.

<sup>b</sup>Indicates that the best cost among the tested algorithms is attained.

## 5. Conclusion

In this paper, we considered the multi-resource generalized assignment problem and proposed a tabu search algorithm in which a sophisticated neighborhood called the chained shift neighborhood is incorporated. It was confirmed through computational comparisons on benchmark instances that the method is effective, especially for types D and E instances, which are known to be very difficult.

## Acknowledgements

The authors are grateful to anonymous referees for valuable comments to improve this paper. This research was partially supported by Scientific Grant-in-Aid by the Ministry of Education, Culture, Sports, Science and Technology of Japan, and by Informatics Research Center for Development of Knowledge Society Infrastructure (COE program of the Ministry of Education, Culture, Sports, Science and Technology, Japan).

## Appendix. Probabilistic analysis

We analyze the probability that a cycle in an improvement graph is subset-disjoint when the assignment  $\sigma(j)$  is chosen from  $\{1, m\}$  uniformly at random for each job  $j \in J$ . For a cycle of length  $l$  ( $l = 2, 3, \dots, m$ ), this probability is

$$1 \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdots \frac{m-l+1}{m} = \frac{m!}{m^l}$$

where  $m^l = m(m-1)(m-2)\cdots(m-l+1)$  is the *falling factorial power*. As shown in [25, 42],

$$\frac{m^l}{m^l} = \begin{cases} 1 + o(1), & \text{if } l \leq m^{1/2-\varepsilon}, \\ e^{-1/2} + o(1), & \text{if } l = m^{1/2}, \\ o(1), & \text{if } l \geq m^{1/2+\varepsilon} \end{cases}$$

holds for any small constant  $\varepsilon > 0$ , where  $o(1) \rightarrow 0$  as  $m \rightarrow +\infty$ .<sup>5</sup> That is, a cycle of length less (resp., more) than  $\sqrt{m}$  tends to be subset-disjoint (resp., not subset-disjoint) if  $m$  is sufficiently large.

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<sup>5</sup> Actually, the middle case is not explicitly shown in [25, 42], but the proof is immediate by a similar argument.

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