Further Extension of the TSP Assign Neighborhood

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Abstract

We introduce a new extension of Punnen's exponential neighborhood for the traveling salesman problem (TSP). In contrast to an interesting generalization of Punnen's neighborhood by De Franceschi, Fischetti and Toth (2005), our neighborhood is searchable in polynomial time, a feature that invites exploitation by heuristic and metaheuristic procedures for the TSP and related problems, including those of De Franceschi, Fischetti and Toth (2005) for the vehicle routing problem.

Key words: traveling salesman problem; local search; exponential neighborhood; assign neighborhood

Local search heuristics are among the main tools to compute near optimal solutions in large instances of combinatorial optimization problems in relatively short time. These heuristics include variants of both constructive algorithms and iterative improvement local search, including more advanced metaheuristic variants such as tabu search, simulated annealing and genetic algorithms, among others (see, e.g., Aarts and Lenstra (1997)). In most cases the neighborhoods used in these algorithms are of polynomial cardinality. One may ask whether it is possible to have exponential size neighborhoods for the traveling salesman problem (TSP) such that the best tour in such a neighborhood can be computed in polynomial time, i.e., exponential size *polynomial time searchable neighborhoods*. Fortunately, the answer to this question is positive. This question is far from being trivial for some generalizations of TSP, e.g., Deineko and Woeginger (2000) conjectured that for the quadratic assignment problem there is no exponential size polynomial time searchable neighborhood.

In this paper we consider the *asymmetric TSP* (called simply TSP in what follows), i.e., the problem of finding a Hamilton cycle (called a *tour*) of minimum total weight in a weighted complete digraph K_n^*. It is easy to adopt notions and results for the asymmetric TSP neighborhoods to the symmetric case. We will always use n as the number of vertices in the complete digraph under consideration and c as the weight function.

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We adopt the definition of a neighborhood for the TSP due to Deineko and Woeginger (2000). Let Π be a set of permutations on n vertices. Then the *neighborhood* with respect to Π of a tour $T = u_1u_2 \dots u_nu_1$ is defined as follows:

$$N_\Pi(T) = \{u_{\pi(1)}u_{\pi(2)} \dots u_{\pi(n)}u_{\pi(1)} : \ \pi \in \Pi\}.$$

While most TSP neighborhoods are of cardinality $2^{\Theta(n)}$ (Deineko and Woeginger (2000), Ergun and Orlin (2005), Gutin, Yeo and Zverovitch (2002)), there are neighborhoods of larger cardinality, $2^{\Theta(n \log\log n)}$ (Burkard, Deineko and Woeginger (1998)) and even $2^{\Theta(n\log n)}$ (Gutin (1984), Punnen (2001), Sarvanov and Doroshko (1981)). We call neighborhoods of cardinality $2^{\Theta(n\log n)}$ *factorial* (note that even $n!$ is encompassed by the order bound of $2^{\Theta(n\log n)}$). The first factorial neighborhood was introduced by Sarvanov and Doroshko (1981) and, independently, by Gutin (1984). Following Deineko and Woeginger (2000), this neighborhood is called the *Assign Neighborhood*; its cardinality is $\lfloor n/2 \rfloor !$. Punnen (2001) introduced an extension of the Assign Neighborhood, which we describe in the next section.

A modification of Punnen's approach led De Franceschi, Fischetti and Toth (2005) to a highly successful local search heuristic for the distance-constrained capacitated vehicle routing problem. The heuristic in De Franceschi, Fischetti and Toth (2005) is not the first local search algorithm that uses exponential cardinality neighborhoods (for other heuristics see, e.g., Ahuja, Orlin and Sharma (2003), Balas and Simonetti (2001)), but it appears to be one of the most successful such heuristics. Also, the heuristic in De Franceschi, Fischetti and Toth (2005) is the first practical heuristic based on a factorial neighborhood.

De Franceschi, Fischetti and Toth (2005) generalize Punnen's neighborhood to a non-polynomial time searchable neighborhood using an Integer Linear Programming solver to find a tour that is not necessarily best in its domain. By contrast, we extend Punnen's neighborhood to a polynomial time searchable neighborhood which we call the Cascade Assign Neighborhood (CAN). When at least one path P_i (see the next section) does not consist of a single vertex, the CAN strictly contains Punnen's neighborhood. Our approach is different from previous ones, see Deineko and Woeginger (2000), Ergun and Orlin (2005), and Gutin, Yeo and Zverovitch (2002).

It should be noted that, in this paper, we only consider Punnen's basic neighborhood. This neighborhood has some relatively straightforward variations (one is described in Punnen (2001)). The variations can also be extended by the CAN construction allowing us to have neighborhoods containing their Punnen's counterparts. While some of the variations may well be of interest in practical applications (such a variation is used in De Franceschi, Fischetti and Toth (2005)), their practical value can only be assessed in computational experiments that are outside the scope of this short paper, which restricts attention to Punnen's basic neighborhood and, thus, the basic CAN.

1 Cascade Assign Neighborhood

Let $C = x_1x_2 \dots x_kx_1$ be a cycle in K_n^*. The operation of *removal* of a vertex x_i $(1 \le i \le k)$ results in the cycle $x_1x_2 \dots x_{i-1}x_{i+1} \dots x_kx_1$. Let $P = y_1y_2 \dots y_q$ be a path of K_n^* with no common vertices with C. The operation of *insertion* of P into an arc (x_j, x_{j+1}) results in the cycle $x_1x_2 \dots x_jy_1y_2 \dots y_qx_{j+1} \dots x_kx_1$. The *cost* of the insertion is defined as

$$c(P, C, j) = c(x_j, y_1) + c(y_q, x_{i+1}) - c(x_j, x_{j+1}).$$

Let $W = [w_{i,j}]$ be an $m \times \ell$-matrix of reals with $m \le \ell$. The *assignment problem* is the problem of finding a one-to-one mapping (*injection*) $\rho : \{1, 2, \dots, m\} \to \{1, 2, \dots, \ell\}$ such that

$\sum_{i=1}^{m} w_{i,\rho(i)}$ is minimal. It is well known that the assignment problem can be solved in time $O(\ell^3)$ (see, e.g., Bang-Jensen and Gutin (2000)).

Now we are ready to describe Punnen's neighborhood of a tour $T = u_1u_2 \dots u_nu_1$. We will use the operation of removal defined earlier. Choose a subset Z of $\{u_1, u_2, \dots, u_n\}$ and, by removing the vertices of Z from T one by one (in any order), form a cycle $C = v_1v_2 \dots v_kv_1$. Partition Z into $s(\leq k)$ sets $Z_1, Z_2, \dots Z_s$ and choose a Hamilton path P_i in the subgraph of K_n^* induced by Z_i for each $1 \leq i \leq s$. *Punnen's neighborhood* of T is formed by tours obtained by choosing an injection $\rho : \{1, 2, \dots, s\} \rightarrow \{1, 2, \dots, k\}$ and inserting each P_i into the arc $(v_{\rho(i)}, v_{\rho(i)+1})$, $1 \leq i \leq s$. Here $v_{k+1} = v_1$. Observe that the weight of such a tour equals the weight of C plus the weight of all $P_i's$ plus the sum of all insertion costs $c(P_i, C, \rho(i))$.

Since the weight of C and the weight of each P_i are constants, to compute a minimum weight tour in Punnen's neighborhood, it suffices to find an injection ρ that minimizes $\sum_{i=1}^{s} c(P_i, C, \rho(i))$. Such an injection can be obtained as a solution to the assignment problem with the $s \times k$ matrix with entries $c(P_i, C, j)$, $1 \leq i \leq s$, $1 \leq j \leq k$. Thus, a minimum weight tour in the neighborhood can be found in time $O(k^3)$.

Consider an example of Punnen's neighborhood. Let $T = 12345671$. Set $Z = \{1, 2, 4, 5\}$, then $C = 3673$. Let $P_1 = 12$ and $P_2 = 54$. Punnen's neighborhood of T has six tours: $3F6G7H3$, where $\{F, G, H\} = \{12, 54, \emptyset\}$.

De Franceschi, Fischetti and Toth (2005) generalized Punnen's neighborhood by creating a large number of intersecting paths from vertices of Z, choosing (optimally or near optimally, using an Integer Linear Programming solver) from this collection a set of paths whose vertices partition Z, and optimally inserting the chosen paths into arcs of C. De Franceschi, Fischetti and Toth (2005) stress that while they avoid fixing, in the beginning, the set of chosen paths, they were unable to avoid inserting *at most one* chosen path into an arc of C.

To alleviate this limitation, we allow paths P_i to be inserted into each other. The operation of *insertion* of $P_i = y_1y_2 \dots y_q$ into an arc (x_t, x_{t+1}) of $P_j = x_1x_2 \dots x_k$ $(i \neq j)$ results in the path $x_1x_2 \dots x_ty_1y_2 \dots y_qx_{t+1} \dots x_k$. The *cost* of the insertion is defined as

$$c(P_i, P_j, t) = c(x_t, y_1) + c(y_q, x_{t+1}) - c(x_t, x_{t+1}).$$

To construct our neighborhood, which we call the *cascade assign neighborhood (CAN)*, of a tour $T = u_1u_2 \dots u_nu_1$ we first construct a cycle $C = v_1v_2 \dots v_kv_1$ and paths $P_1, P_2, \dots, P_s$ as above (in what follows we denote $P_0 = C$). Now we do not assume that $s \leq k$, instead we assume that, for each $i \leq s$, the total number of arcs in $P_0, P_1, \dots, P_{i-1}$ is at least i. A tour in the CAN is formed by inserting each P_i into an arc of some P_j $(0 \leq j < i)$ such that no two paths are inserted into the same arc. The weight of this tour equals the weight of all $P_i's$ plus the sum of the costs of all s insertions. Notice that, unlike the form of all tours in Punnen's neighborhood, the form of many tours in the CAN depends on the order of paths $P_1, P_2, \dots, P_s$.

Since the weight of each P_i is a constant, to compute a minimum weight tour in the CAN, it suffices to find an optimal sequence of the injections. Let $a = (w_t, w_{t+1})$ be an arc of P_j, $0 \leq j \leq s$. Then define the *cost* of insertion of P_i into a to be equal to $c(P_i, P_j, t)$ if $i > j$ and to ∞, otherwise. Let $a_1, a_2, \dots, a_\alpha$ be a sequence of all arcs of $P_0, P_1, \dots, P_s$. Then we can find an optimal sequence of the injections, by solving the assignment problem with the $s \times \alpha$ matrix with entries $m_{i,q}$, where $m_{i,q}$ is the cost of insertion of P_i, $1 \leq i \leq s$, into the arc a_q. Thus, an optimal tour in the CAN can be found in time $O(\alpha^3) = O(n^3)$.

Consider the example that we have earlier used for Punnen's neighborhood. Let $T = 12345671$. Let $P_0 = 3673$, $P_1 = 12$ and $P_2 = 54$. Apart from the six tours of Punnen's neighborhood of T listed above, the CAN neighborhood of T has three other tours: $3Q673, 36Q73, 367Q3$, where $Q = 1542$.

2 How Large Can a TSP Neighborhood Be?

Let p_i be the number of vertices in P_i, $0 \leq i \leq s$. Form a vector $p = (p_0, p_1, \ldots, p_s)$ and observe that the CAN has exactly $\mathrm{can}(p) = p_0(p_0 + p_1 - 2) \cdots (p_0 + p_1 + \cdots + p_s - 2s)$ tours. Let $p^i = (p_0, p_1, \ldots, p_{i-2}, p_{i-1} + p_i - 1, 1, p_{i+1}, \ldots, p_s)$. Observe that if $p_i > 1$ for some $i \geq 1$, then $\mathrm{can}(p^i) > \mathrm{can}(p)$. Thus, for a fixed s, the CAN of maximum cardinality has the vector $p = (n - s, 1, 1, \ldots, 1)$. This means that the maximum cardinality of the CAN equals the maximum cardinality of Punnen's neighborhood, which was computed by Gutin (1999).

It was shown by Gutin (1999) that, for each integer $d \geq 0$, by combining a polynomial number of Punnen's neighborhoods with vector $p = (n - s, 1, 1, \ldots, 1)$ for optimal s, one can obtain a polynomial time searchable TSP neighborhood of cardinality $\Theta(\lfloor \frac{n+1}{2} \rfloor ! e^{\sqrt{n/2}} n^{d-q})$, where $q = 1/4$ for even n and $q = 3/4$ for odd n. Gutin, Yeo and Zverovitch (2002) pose the question of whether there exist polynomial time searchable neighborhoods of larger cardinality. In Section 1, we mentioned a variation of Punnen's neighborhood described in Punnen (2001). Unfortunately, this variation, even extended to the CAN, does not lead us to larger neighborhoods (see Punnen (2001)).

Deineko and Woeginger (2000) conjectured that, for some $\beta > 1/2$ there exists a polynomial time searchable neighborhood of cardinality at least $\lfloor \beta(n-1) \rfloor !$. Of relevance to this conjecture, Gutin and Yeo (2003) proved that there is no polynomial time searchable neighborhood of cardinality at least $(n-k)!$ for each integral constant k unless NP⊆P/poly (similarly to P=NP, NP⊆P/poly is highly unlikely to hold). The idea that defines P/poly is that, for each input size n, one is able to compute a polynomial-sized "key for size n inputs." It is allowed that the computation of this key may take time exponential in n (or worse). P/poly means solvable in polynomial time (in input size n) / given the key for inputs of size n. For formal definitions of P/poly and related nonuniform complexity classes, consult Balcazar, Diaz and Gabarro (1995).

3 Conclusions

Finally, we observe that the Cascade Assign Neighborhood effectively operates as a collection of ejection chain moves, in which paths are permitted to replace (eject) arcs, subject to special limitations. (In the present case, ejections are only transmitted in a single direction along the hierarchy created by the path indexing.) The effectiveness of ejection chain methods for the TSP, as demonstrated by the studies of Rego and Glover (2002) and Gamboa, Rego and Glover (2005), suggest that our current procedure may also be used to advantage in combination with such ejection chain constructions. Similarly, our procedure can be used to supplement the appealing vehicle routing approach of De Franceschi, Fischetti and Toth. These possibilities open up a wide range of future avenues for research.

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