Tabu Search for the Graph Coloring Problem (extended)

Enrico Malaguti, Paolo Toth

Dipartimento di Elettronica, Informatica e Sistemistica, University of Bologna
Viale Risorgimento, 2 - 40136 - Bologna (Italy)
e-mails: `enrico.malaguti@unibo.it, paolo.toth@unibo.it`

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Given an undirected graph $G = (V, E)$, the *Graph Coloring Problem* (GCP) requires to assign a color to each vertex in such a way that colors on adjacent vertices are different and the number of colors used is minimized. A subset of V is called a *stable set* if no two adjacent vertices belong to it. A k *coloring* of G is a coloring which uses k colors, and corresponds to a partition of V into k stable sets.

The Graph Coloring problem is in the original list of NP-hard problems (see Garey and Johnson [8]), and has received a large attention in the literature, not only for its real world applications in many engineering fields, including, among many others, scheduling [10], timetabling [4], register allocation [3], train platforming [2], frequency assignment [7] and communication networks [14], but also for its theoretical aspects and for its difficulty from the computational point of view.

Recently, Malaguti et al. [11] proposed an evolutionary algorithm combining a Tabu Search procedure, based on the *Impasse Class Neighborhood*, and a crossover operator.

The *Impasse Class Neighborhood* was defined by Morgenstern in [13]. It is a structure used to improve a partial k *coloring* to a complete coloring of the same value, thus, a method which works with a fixed number of colors k and partial colorings (not all vertices are colored). A solution S is a partition of V in $k+1$ color classes $\{V_1, ..., V_k, V_{k+1}\}$ in which all classes, but possibly the last one, are stable sets. This means that the first k classes constitute a partial feasible k *coloring*, while all vertices that do not fit in the first k classes are in the last one. Making this last class empty gives a complete feasible k *coloring*. To move from a solution S to a new solution S' in the neighborhood, one can choose an uncolored vertex $v \in V_{k+1}$, assign v to a different color class, say h, and move to class $k+1$ all vertices v' in class h that are adjacent to v. This ensures that color class h remains feasible.

While Morgenstern embedded the neighborhood in a simulated annealing algorithm, Malaguti et al.[11] exploited it in a Tabu search algorithm. The uncolored vertex v is randomly chosen, while the class h is chosen by comparing different target classes by means of an evaluating function $f(S)$. Rather than simply minimizing $\mid V_{k+1} \mid$, the algorithm minimizes the global degree of the uncolored vertices:

$$f(S) = \sum_{v \in V_{k+1}} \delta(v) \tag{1}$$

where $\delta(v)$ represents the degree of vertex v. This choice forces vertices having small degree, which are easier to color, to enter class $k+1$. To avoid cycling, the authors use the following tabu rule: a vertex v cannot take the same color h it took at least one of the last T iterations; for this purpose a tabu list stores the pair (v, h). While pair (v, h) remains in the tabu list, vertex v cannot be assigned to color class h. In addition, the algorithm implements an *Aspiration Criterion*: a tabu move can be performed if it improves on the best solution encountered so far.

Two Tabu Search algorithms based on the *Impasse Class Neighborhood*, called DYN-PARTIALCOL and FOO-PARTIALCOL, were proposed by Blöchliger and Zufferey [1]. Differently from the Tabu Search in [11], which selects at random the next vertex to color, DYN-PARTIALCOL and FOO-PARTIALCOL, select as next vertex to color the one that, entering the best color class, produces the best solution in the neighborhood. This approach increases the size of the neighborhood reducing at the same time the randomness introduced in the search. Thus, to avoid premature convergence, the authors use an evaluating function that simply minimizes $|V_{k+1}|$.

In the evolutionary algorithm by Malaguti et al. [11], as well as in the Tabu Search algorithms by Blöchliger and Zufferey [1], one of the crucial component for the overall algorithm efficacy is the *Impasse Class Neighborhood* embedded within a Tabu Search scheme. In particular the algorithm by Malaguti et al., integrated with a post-optimization procedure, can find 30 times the best known solution for a set of 34 hard graph coloring instances from the DIMACS benchmark, and in two cases it is the only algorithm in the literature able to compute the best solution. The computing times of the algorithm range from few seconds for small graphs to approximately three hours for very large graphs (1000 vertices).

Among the many algorithms proposed for the graph coloring problem, we have to mention the Tabu Search procedure by Hertz and de Werra [9], which was one of the first metaheuristic algorithms proposed for the problem. The algorithm, called TABUCOL, considers a fixed number k of available colors, and moves among complete colorings. A solution is thus represented by a k coloring of the graph, where some edges are conflicting, i.e., both endpoints share the same color. A move consists in changing the color of one vertex, and the evaluation function measures the number of conflicts in the current coloring. The algorithm considers only moves involving critical vertices, i.e., vertices whose color is currently in conflict with some adjacent vertex. A tabu list stores the assignment of colors to vertices, and forbids this assignment to be replicated for a specified number of iterations. TABUCOL is a well known algorithm which was successfully improved and embedded as a procedure in more complex algorithms (see, e.g. Dorne and Hao [5] and Galinier and Hao [6]).

The reader is referred to the recent survey by Malaguti and Toth [12] for a detailed review of algorithms and computational results for the graph coloring problem.

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